Exactly Solving a class of QCQPs via Semidefinite Relaxation with Bipartite Sparsity Patterns

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QCQP: Quadratic Constrained Quadratic Programming

Let $Q^0, \ldots, Q^m$ be $n \times n$ symmetric matrices, $\boldsymbol{b} \in \mathbb{R}^m$.

$$\begin{aligned} v^* := \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p, \quad p \in [m] := \{1, \ldots, m\}, \end{aligned} \qquad (\mathcal{P})$$

Applications

MAX-CUT, sensor location problems, optimal flow problems,...

- ($\mathcal{P}$) is a homogeneous form (no linear terms).
- Calculation of v^* is NP-hard.

Semidefinite Programming (SDP) Relaxation

$A \bullet B$: Frobenius (component-wise) inner product of A and B

$X \succeq O$: Positive semidefinite

$$
\begin{aligned}
v^* &= \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \,\middle|\, \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p \quad \forall p \in [m] \right\} \\
&= \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} \\
&\ge \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X \succeq \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} \qquad (\mathcal{P}_{\mathrm{R}}) \\
&=: v^*_{\mathrm{SDP}}
\end{aligned}
$$

v^*_{SDP} can be caluculated in polynomial time.

Exactness of SDP Relaxation

$$v^* \geq \min \left\{ Q^0 \bullet X \;\middle|\; \begin{array}{l} X \succeq O \\ Q^p \bullet X \leq b_p \quad \forall p \in [m] \end{array} \right\} = v^*_{\mathrm{SDP}}$$

$=$ holds

$\iff$ SDP relaxation is exact

Interested in

What conditions of QCQPs guarantee the exact SDP relaxation?

$(v^* = v^*_{\mathrm{SDP}})$

$\implies$ classify QCQPs as tractable or not.

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$=$ holds

$\iff$ SDP relaxation is exact

$\iff$ an optimal solution X^* of rank-1 exists

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What conditions of QCQPs guarantee the exact SDP relaxation?

$(v^* = v^*_{\mathrm{SDP}})$

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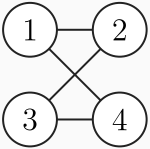
Outline

- Motivation
- Sparsity structures of QCQPs
- Exactness condition for bipartite sparsity structures
- Example
- Sign-definite QCQPs
- Comparison with existing research
- Summary

Aggregated Sparsity Pattern of QCQP: $G(\mathcal{V}, \mathcal{E})$

$$\mathcal{V} := [n],$$

$$\mathcal{E} := \left\{ (i,j) \in \mathcal{V}^2 \;\middle|\; i \neq j,\ Q^p_{ij} \neq 0 \text{ for some } p \in \{0, \dots, m\} \right\}.$$



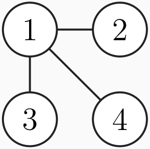
ex. $\min \{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \}$

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

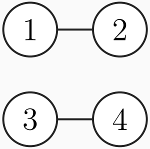
Forest and Bipartite Graph

Let $G(\mathcal{V}, \mathcal{E})$ be a nonempty graph.

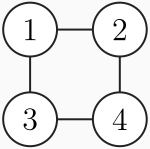
	Cycle		
G	Odd length	Even length	#Components
Tree			1
Forest			≥ 1
Bipartite		✓	≥ 1



Tree



Forest



Bipartite

Dual of SDP Relaxation

$$v^*_{\mathrm{DSDP}} \;:=\; \max\left\{ -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \;\middle|\; \boldsymbol{y} \geq 0,\; S(\boldsymbol{y}) \succeq O \right\} \qquad (\mathcal{D}_{\mathrm{R}})$$

where matrix function over $\mathbb{R}^m$:

$$S(\boldsymbol{y}) := Q^0 + \sum_{p=1}^{m} y_p Q^p.$$

When a given QCQP has the sparsity pattern $G(\mathcal{V}, \mathcal{E})$,

- $S(y)_{k\ell} = 0$ if $(k, \ell) \notin \mathcal{E}$.
- $\implies$ $S(y)$ follows the same sparsity $\mathcal{E}$.

Assumption in This Talk

Assumption

(i) Both $(\mathcal{P}_{\mathrm{R}})$ and $(\mathcal{D}_{\mathrm{R}})$ have optimal solutions, and

(ii) At least one of the following two conditions holds:

(a) the feasible region of $(\mathcal{P}_{\mathrm{R}})$ is bounded, or

(b) the set of optimal solutions for $(\mathcal{D}_{\mathrm{R}})$ is bounded.

- strong duality holds: (Kim and Kojima[1])

  $\exists(X^*, \boldsymbol{y}^*)$: solutions of $(\mathcal{P}_{\mathrm{R}})$ and $(\mathcal{D}_{\mathrm{R}})$ such that

$$X^* S(\boldsymbol{y}^*) = O.$$

[1] Sunyoung Kim and Masakazu Kojima. *Strong duality of a conic optimization problem with a single hyperplane and two cone constraints.* arXiv:2111.03251v2. 2021.

Sufficient Conditions for Exactness

$v^* = v^*_{\mathrm{SDP}}$

$\iff$ ($\mathcal{P}_{\mathrm{R}}$) has an optimal X^* satisfying $\mathrm{rank}(X^*) \leq 1$

$\Longleftarrow$ ($\mathcal{D}_{\mathrm{R}}$) has an optimal $\boldsymbol{y}^*$ satisfying $\mathrm{rank}\{S(\boldsymbol{y}^*)\} \geq n-1$

(under the strong duality)

$\Longleftarrow$ $\mathrm{rank}\{S(\boldsymbol{y})\} \geq n-1 \quad \forall \boldsymbol{y} \geq \mathbf{0}$ satisfying $S(\boldsymbol{y}) \succeq O$

$$\Longleftarrow \begin{cases} S(\boldsymbol{y}) \succeq O, \ S(\boldsymbol{y})\mathbf{1} > \mathbf{0}, \\ S(\boldsymbol{y})_{ij} > 0 \ \forall (i,j) \in \mathcal{E} \end{cases}$$ [Grone et al., 1992]

Since $\boldsymbol{y} \geq 0, \ S(\boldsymbol{y}) \succeq O, \ S(\boldsymbol{y})_{ij} \leq 0$ has no solutions,

we conclude $S(\boldsymbol{y})_{ij} > 0 \ \forall (i,j) \in \mathcal{E}$.

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[Grone et al., 1992]

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New Exactness Condition

Suppose G is a bipartite.

Exactness Condition for QCQPs with Bipartite Structures

For all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0, \ S(\boldsymbol{y}) \succeq O, \ [S(\boldsymbol{y})]_{k\ell} \leq 0, \tag{1}$$

- Checking $|\mathcal{E}|$ feasibility systems is required.
- (1) is a SDP, i.e., tractable.

Example 2

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^2 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^3 \boldsymbol{x} \leq 5 \end{aligned}$$

where

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \; Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

Sparsity of Example 2

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

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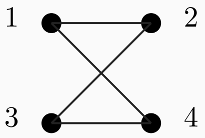


$$\mathcal{E} = \left\{ \begin{array}{l} (1,2),(2,1),(1,4),(4,1), \\ (2,3),(3,2),(3,4),(4,3) \end{array} \right\}.$$

Systems of Example 2

Consider the problem for $(k,\ell) \in \mathcal{E}$:

$$\begin{aligned} \mu^* = \min \quad & S(\boldsymbol{y})_{k\ell} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \boldsymbol{0},\ S(\boldsymbol{y}) \succeq O. \end{aligned} \tag{2}$$

(k,ℓ)	$(1,2)$	$(2,3)$	$(1,4)$	$(3,4)$
μ^*	18.58	12.84	8.897	0.3215

All positives $\implies$ the following systems have no solutions:

$$\boldsymbol{y} \geq \boldsymbol{0},\ S(\boldsymbol{y}) \succeq O,\ S(\boldsymbol{y})_{k\ell} \leq 0. \tag{1}$$

Comparison of Exactness Conditions

	Graph G	Systems to check
Burer & Ye[2]	no edges	$\mathcal{S}_=$ for all (k,ℓ) such that $k=\ell$
Azuma et al.[3]	forest	$\mathcal{S}_=$ for all $(k,\ell)\in\mathcal{E}$
Proposed method	bipartite	$\mathcal{S}_\leq$ for all $(k,\ell)\in\mathcal{E}$

where systems are:

a LP or a SDP: tractable problem

$$\text{find } \boldsymbol{y}\geq 0 \text{ such that } S(\boldsymbol{y})\succeq O,\ [S(\boldsymbol{y})]_{k\ell}\diamondsuit 0. \qquad (\mathcal{S}_\diamondsuit)$$

[2] Samuel Burer and Yinyu Ye. "Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs". In: *Mathematical Programming* 181.1 (2020), pp. 1–17.

[3] Godai Azuma et al. "Exact SDP Relaxations of Quadratically Constrained Quadratic Programs with Forest Structures". In: *Journal of Global Optimization* 82.2 (2022), pp. 243–262.

Sign-definite QCQP

Definition of Sign-definite QCQP

- For all (i, j), the set $T_{ij} := \{Q_{ij}^0, \ldots, Q_{ij}^m\}$ is sign-definite
- QCQPs with no sparsity

all nonnegative or all nonpositive

Theorem 2[4] [Sojoudi and Lavaei, 2014]

$$\prod_{(i,j)\in\mathcal{C}} \sigma_{ij} = (-1)^{|\mathcal{C}|} \quad \text{for all cycles } \mathcal{C} \text{ in } G \tag{3}$$

where

$$\sigma_{ij} = \begin{cases} +1 & (T_{ij} \text{ has all nonnegative}), \\ -1 & (T_{ij} \text{ has all nonpositive}). \end{cases}$$

[4] Somayeh Sojoudi and Javad Lavaei. "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure". In: *SIAM Journal on Optimization* 24.4 (2014), pp. 1746–1778.

Proposed Condition Covers Condition (3)

Proposition

If a given $(\mathcal{P})$ satisfies the condition (3),
proposed condition can detect the exactness of its SDP relaxation.

Idea:

We develop conversion method of QCQPs such that

- The obtained QCQP has bipartite sparsity.
- The obtained QCQP satisfies proposed condition:
  $\forall(k, \ell) \in \mathcal{E}$, the system (1) has no solutions.

Summary

Summary

- QCQPs whose G is bipartite were analyzed.
- New sufficient condition of $v^* = v^*_{\mathrm{SDP}}$ was proposed.
- It was compared with three existing results.

Future works

- Approximated problems of QCQPs with exact SDP relaxation
- Analysis of QCQPs transformed from general problems

More information is available at arXiv:2204.09509,

"Exact SDP relaxations for quadratic programs with bipartite graph structures."

Thank you for your attention!

Backup Slides

Cycle Basis

$\mathcal{C} := \{\mathcal{C}_1, \dots, \mathcal{C}_\kappa\}$: set of (simple) cycles

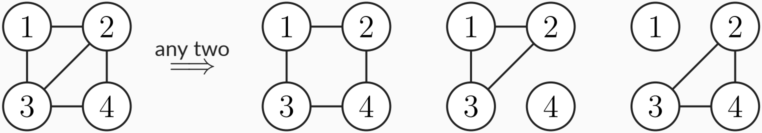
$A \triangle B$: symmetric difference of A and B

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

$\mathcal{C}$ is called a cycle basis of G

$\iff$
- ∘ allows any cycle in G to be expressed by $\triangle$ of its elements
- ∘ be a minimum set

Ex.



Instance for SDP relaxation (QCQP side)

Example 1[5]

$$\begin{aligned} v^* = \min \quad & x^2 + y^2 \\ \text{s.t.} \quad & y^2 \geq 1,\ x^2 - xy \geq 1,\ x^2 + xy \geq 1 \end{aligned}$$

From last two inequality,

- $$\left.\begin{aligned} xy > 0 &\implies xy > -xy \\ xy < 0 &\implies -xy > xy \end{aligned}\right\} \implies x^2 \geq |x||y| + 1$$

- $$x^2 \geq 1.$$

$$\therefore \quad x^2 + y^2 \geq (|x||y| + 1) + 1 \geq 3$$

[5] **Luo-Lecture14**.

Instance for SDP relaxation (SDP side)

$$
\begin{aligned}
v^* &= \min\left\{x^2+y^2 \,\middle|\, y^2 \ge 1,\ x^2-xy \ge 1,\ x^2+xy \ge 1\right\}\\
&= \min\left\{\boldsymbol{x}^{\mathrm{T}} I \boldsymbol{x} \,\middle|\, \begin{array}{l} \boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\boldsymbol{x} \ge 1,\\ \boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}1 & -1/2\\ -1/2 & 0\end{bmatrix}\boldsymbol{x} \ge 1,\ \boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}1 & 1/2\\ 1/2 & 0\end{bmatrix}\boldsymbol{x} \ge 1 \end{array}\right\}\\
&\ge \min\left\{I \bullet X \,\middle|\, \begin{array}{l} \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \bullet X \ge 1,\ X \succeq O\\ \begin{bmatrix}1 & -1/2\\ -1/2 & 0\end{bmatrix} \bullet X \ge 1,\ \begin{bmatrix}1 & 1/2\\ 1/2 & 0\end{bmatrix} \bullet X \ge 1 \end{array}\right\}\\
&= \min\left\{X_{11}+X_{22} \,\middle|\, X_{22} \ge 1,\ X_{11}-X_{12} \ge 1,\ X_{11}+X_{12} \ge 1,\ X \succeq O\right\}
\end{aligned}
$$

$X = I$ is feasible $\implies$ $v^*_{\mathrm{SDP}} \le 2 < 3 \le v^*$

QCQP's Applications

Used in various problems:

- MAX-CUT, MAX-CLIQUE
- sensor (facility) location problem, pooling problem
- optimal flow problem, polynomial optimization
- (robust / sparse) principal component analysis, phase retrieval

Alternative Result for Theorem 3.7

Theorem 3.7'

Suppose Assumption 3.8 holds & G is a forest and connected graph. Then, $v^* = v^*_{\mathrm{SDP}}$ if for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0, \; S(\boldsymbol{y}) \succeq O, \; [S(\boldsymbol{y})]_{k\ell} = 0, \qquad \text{(??)}$$

Estimation of Rank of Sparse Matrices

Let $G([n], \mathcal{E}_M)$ be the sparsity pattern graph of a matrix M.

For tree (connected forest) [Johnson et al., 2003, Corollary 3.9]

$$\left.\begin{array}{l} G \text{ is connected and forest} \\ M \succeq O \\ M_{ij} \neq 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n-1$$

For bipartite [Grone et al., 1992, Proposition 1]

$$\left.\begin{array}{l} G \text{ is connected and bipartite} \\ M \succeq O,\ M\mathbf{1} > 0 \\ M_{ij} > 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n-1$$

where $\mathbf{1}$ is the one vector $[1 \ \cdots \ 1]^{\mathrm{T}}$.

Rank of Solutions of Dual SDP Relaxation

Assume the strong duality holds.

Lemma **[Burer and Ye, 2020]**

$(\mathcal{D}_{\mathrm{R}})$ has a solution of rank $(n-1)$ or more $\implies v^* = v^*_{\mathrm{SDP}}$

Proof.

Let $(X^*, \boldsymbol{y}^*)$ be a solution satisfying $X^* S(\boldsymbol{y}^*) = O$.
From Sylvester's rank inequality,

$$\mathrm{rank}(X^*) \leq n - \mathrm{rank}\{S(\boldsymbol{y}^*)\} + \mathrm{rank}\{X^* S(\boldsymbol{y}^*)\}$$

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Exactness can be considered from the dual side.

Exactness Condition on Existing Research

Non-homogeneous version of QCQP:

$$\begin{aligned} v^* \;:=\; \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} + 2(\boldsymbol{q}^0)^{\mathrm{T}} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} + 2\boldsymbol{q}^{p\mathrm{T}} \boldsymbol{x} \le b_p, \quad p \in [m], \end{aligned}$$

Define $S(y) = Q^0 + \sum y_p Q^p, \quad s(y) = q_0 + \sum y_p q_p$.

Exactness Condition for Diagonal QCQPs **[Burer and Ye, 2020]**

Suppose $Q^0, Q^1, \ldots, Q^m$ are diagonal matrices.
If for all $k \in \{1, \ldots, n\}$, the following system is infeasible:

$$y \ge 0,\ S(y) \succeq 0,\ [S(y)]_{kk} = 0,\ [s(y)]_k = 0.$$

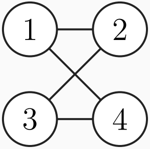
the SDP relaxation is exact.

Relationship between $S(\boldsymbol{y})$ and G

Observation

(i,j) element

$$S(\boldsymbol{y})_{ij} = 0 \quad \forall \boldsymbol{y} \in \mathbb{R}^m \quad \forall (i,j) \notin \mathcal{E}$$



$$\because \quad S(\boldsymbol{y})_{ij} = Q^0_{ij} + y_1 Q^1_{ij} + \cdots + y_m Q^m_{ij}$$

$\Longrightarrow$ $S(\boldsymbol{y})$ has the same sparsity structure as that of QCQP.

ex.

$$S(\boldsymbol{y}) = \begin{bmatrix} -y_1 & -2+y_1 & 0 & 2 \\ -2+y_1 & +4y_1 & -1-y_1 & 0 \\ 0 & -1-y_1 & 5+6y_1 & 1+y_1 \\ 2 & 0 & 1+y_1 & -4-2y_1 \end{bmatrix}$$

Ex. $n = 2, \quad m = 1,$

$$\min. \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} \boldsymbol{x} \qquad \text{s.t.} \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \boldsymbol{x} \leq 1.$$

- $\mathcal{E} = \{(1,2), (2,1)\}$
- Systems — only for $(k, \ell) = (1, 2)$

$$y_1 \geq 0, \quad \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} + y_1 \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \succeq O, \quad -1 + 4y_1 \leq 0$$

- $S(y) \succeq O \iff y_1 \geq 3 + \frac{3\sqrt{6}}{2} \simeq 6.67$

second inequality $\implies -1 + 4y_1 > 0 \qquad \implies v^* = v^*_{\mathrm{SDP}}$

No solutions

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v^* = \min \left\{ \boldsymbol{x}^\mathrm{T} Q^0 \boldsymbol{x} \mid \boldsymbol{x}^\mathrm{T} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P})$$

$$\downarrow$$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^\mathrm{T} \left(Q^0 + \varepsilon P \right) \boldsymbol{x} \mid \boldsymbol{x}^\mathrm{T} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P}^\varepsilon)$$

- $(\mathcal{P}^\varepsilon)$ converges to $(\mathcal{P})$ as $\varepsilon \downarrow 0$.
- If $P \preceq O$, then $v_\varepsilon^* \leq v^*$.

ε-Perturbed QCQPs

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$$v^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}$}$$

$$\downarrow$$

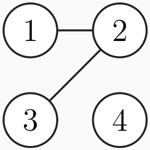
$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}^\varepsilon$}$$

Preferred edges can be added to G

- ($\mathcal{P}^\varepsilon$) converges to ($\mathcal{P}$) as $\varepsilon \downarrow 0$.
- If $P \preceq O$, then $v_\varepsilon^* \leq v^*$.

Instance of Adding Edges

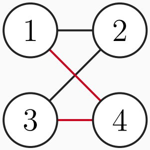
$$\min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 \qquad \right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \right\}$$



$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

Instance of Adding Edges

$$\min\left\{\, \boldsymbol{x}^{\mathrm{T}}\left(Q^0 + \varepsilon P\right)\boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \,\right\}$$



$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Perturbation and Exactness

$\{\varepsilon_t\}_{t=1}^{\infty} \subseteq \mathbb{R}_+$: monotonically decreasing and $\lim_{t\to\infty} \varepsilon_t = 0$

$P \neq O : n \times n$ negative semidefinite matrix

Lemma 4.1, Lemma 4.4, and Lemma 4.5

Suppose Assumption 3.6 holds, or
Assumption 3.8 and an additional one hold.

SDP relaxation of $(\mathcal{P}^{\varepsilon})$ is exact for all $\varepsilon = \varepsilon_1, \varepsilon_2, \ldots$ $\implies v^* = v^*_{\mathrm{SDP}}$.

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of ($\mathcal{P}$).

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest and connected.

Then, $v^* = v^*_{\text{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \tag{??}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite and connected.

Then, $v^* = v^*_{\text{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \tag{1}$$

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of ($\mathcal{P}$).

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \qquad \text{(??)}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \qquad (1)$$

Sketch of Proof

Proof of Theorem 4.6.

1. Let $\mathcal{F}$ be the set of additional edges.
2. Define $P \preceq O$ as

$$P_{ij} = \begin{cases} -\deg(i) & \text{if } i = j, \\ 1 & \text{if } (i,j) \in \mathcal{F} \text{ or } (j,i) \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

3. ($\mathcal{P}^{\varepsilon}$) satisfies assumptions of Theorem 3.10.
   $\implies$ SDP relaxation of ($\mathcal{P}^{\varepsilon}$) is exact.
4. Using Lemma 4.4 (4.5), we conclude $v^* = v^*_{\mathrm{SDP}}$. □

Previous Works

- Trust-region subproblems (TRS: QCQP with one constraint)
  Yakubovich[1971]
- Extended TRS (TRS + linear constraints)
  Jeyakumar[2014], Hsia and Sheu[2013], Locatelli[2016]
- QCQPs with sign-definiteness
  Kim and Kojima[2003], Sojoudi and Lavaei[2014]

- Exactness by faces of convex lagrangian multipliers
  Wang and Kılınç-Karzan[2021]
- Rank-one generated cones
  Argue, Kılınç-Karzan and Wang[2020]

Essence of Proof (1)

Transform from general QCQPs to sparse QCQPs.

1. Objective function and constraints have the form:

$$\boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} Q^p_{11} & Q^p_{12} & Q^p_{13} & Q^p_{14} \\ Q^p_{21} & Q^p_{22} & Q^p_{23} & Q^p_{24} \\ Q^p_{31} & Q^p_{32} & Q^p_{33} & Q^p_{34} \\ Q^p_{41} & Q^p_{42} & Q^p_{43} & Q^p_{44} \end{bmatrix} \boldsymbol{x}, \quad \forall p,$$

   where $n = 4$, Q^p: symmetric matrices.

2. Assume remove edges $(1, 3)$ and $(2, 4)$ because of $Q^p_{13} < 0, \;\; Q^p_{24} < 0.$

Essence of Proof (2)

3. New variable $\boldsymbol{z} := -\boldsymbol{x}$ is introduced.
4. It can be written as

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{cccc|cccc} Q_{11}^p & Q_{12}^p & 0 & Q_{14}^p & 0 & 0 & -\frac{1}{2}Q_{13}^p & 0 \\ Q_{21}^p & Q_{22}^p & Q_{23}^p & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{32}^p & Q_{33}^p & Q_{34}^p & -\frac{1}{2}Q_{31}^p & 0 & 0 & 0 \\ Q_{41}^p & 0 & Q_{43}^p & Q_{44}^p & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2}Q_{13}^p & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ -\frac{1}{2}Q_{31}^p & 0 & 0 & 0 & & O & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}, \ \forall p.$$

5. Off-diagonal elements are all nonnegative with some zero elements.

The obtained problem satisfies Theorem 4.6.