Exactness conditions for SDP relaxation of bipartite-structured and sign-indefinite QCQPs

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QCQP: Quadratically Constrained Quadratic Programming

Let $Q^0, \ldots, Q^m$ be $n \times n$ symmetric matrices, $\boldsymbol{b} \in \mathbb{R}^m$.

$$\begin{aligned} v^* \ := \ \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p, \quad p \in [m] := \{1, \ldots, m\}, \end{aligned} \tag{$\mathcal{P}$}$$

Applications

MAX-CUT, sensor location problems, optimal flow problems,...

- $(\mathcal{P})$ is a homogeneous form (no linear terms).
- Calculation of v^* is NP-hard.
- Semidefinite programming relaxation is often used.

Semidefinite Programming (SDP) Relaxation

$$
\begin{aligned}
v^* &= \min \left\{ \boldsymbol{x}^\mathrm{T} Q^0 \boldsymbol{x} \,\middle|\, \boldsymbol{x}^\mathrm{T} Q^p \boldsymbol{x} \le b_p \quad \forall p \in [m] \right\} \\
&= \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X = \boldsymbol{x}\boldsymbol{x}^\mathrm{T} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} \\
&\ge \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X \succeq \boldsymbol{x}\boldsymbol{x}^\mathrm{T} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} =: v^*_{\mathrm{SDP}} \qquad (\mathcal{P}_\mathrm{R})
\end{aligned}
$$

where

- $Q^p \bullet X := \sum_{i,j} Q^p_{ij} X_{ij}$,
- $X \succeq O \iff X$ is positive semidefinite.

Pros: v^*_{SDP} can be calculated in polynomial time.

Cons: In general, $v^* \ne v^*_{\mathrm{SDP}}$.

Exactness of SDP Relaxation

$$v^* \geq \min \left\{ Q^0 \bullet X \;\middle|\; \begin{array}{l} X \succeq O \\ Q^p \bullet X \leq b_p \quad \forall p \in [m] \end{array} \right\} = v^*_{\mathrm{SDP}}$$

$=$ holds

$\iff$ SDP relaxation is exact

Interested in

What conditions of QCQPs guarantee the exact SDP relaxation?

$(v^* = v^*_{\mathrm{SDP}})$

$\Longrightarrow$

- Classify QCQPs according to tractability
- Identify the gap between a QCQP and its SDP relaxation

Exactness of SDP Relaxation

$$v^* \geq \min \left\{ Q^0 \bullet X \;\middle|\; \begin{array}{l} X \succeq O \\ Q^p \bullet X \leq b_p \quad \forall p \in [m] \end{array} \right\} = v^*_{\mathrm{SDP}}$$

$=$ holds
$\iff$ SDP relaxation is exact

$\iff$ an optimal solution X^* of rank-1 exists

Interested in

What conditions of QCQPs guarantee the exact SDP relaxation?
$(v^* = v^*_{\mathrm{SDP}})$

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- Classify QCQPs according to tractability
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- ”Full-rank minus one” solutions of dual SDP relaxation
- Idea for diagonal matrices $Q^0, \ldots, Q^m$

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Primal/Dual Problems of SDP Relaxation

$$v^* = \min_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p,\ p \in [m] \right\} \qquad (\mathcal{P})$$

$$\swarrow$$

Primal

$$\begin{aligned} v^*_{\mathrm{SDP}} = \min_{X} \quad & Q^0 \bullet X \\ \text{s.t.} \quad & Q^p \bullet X \leq b_p,\ p \in [m] \\ & X \succeq O \end{aligned} \qquad (\mathcal{P}_R)$$

Dual

$$\begin{aligned} \max_{\boldsymbol{y} \in \mathbb{R}^m} \quad & -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \boldsymbol{0}, \\ & S(\boldsymbol{y}) \succeq O \end{aligned} \qquad (\mathcal{D}_R)$$

where

$$S(\boldsymbol{y}) := Q^0 + \sum_{p=1}^{m} y_p Q^p \quad \text{for } \boldsymbol{y} \in \mathbb{R}^m.$$

Dual Solution of rank-$(n-1)$ is Important for Exactness

Primal

$$v^*_{\mathrm{SDP}} = \min_{X} \quad Q^0 \bullet X \qquad (\mathcal{P}_R)$$
$$\text{s.t.} \quad Q^p \bullet X \le b_p,\ p \in [m]$$
$$X \succeq O$$

Dual

$$\max_{\boldsymbol{y} \in \mathbb{R}^m} \quad -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \qquad (\mathcal{D}_R)$$
$$\text{s.t.} \quad \boldsymbol{y} \ge \boldsymbol{0},$$
$$S(\boldsymbol{y}) \succeq O$$

Exact if

Rank-1 solution X^*

Dual Solution of rank-$(n-1)$ is Important for Exactness

Primal

$$v^*_{\mathrm{SDP}} = \min_{X} \quad Q^0 \bullet X$$
$$\text{s.t.} \quad Q^p \bullet X \leq b_p,\ p \in [m] \qquad (\mathcal{P}_R)$$
$$X \succeq O$$

Dual

$$\max_{\boldsymbol{y} \in \mathbb{R}^m} \quad -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y}$$
$$\text{s.t.} \quad \boldsymbol{y} \geq \boldsymbol{0}, \qquad (\mathcal{D}_R)$$
$$S(\boldsymbol{y}) \succeq O$$

Exact if — Rank-1 solution X^* $\Longleftarrow$ Rank-$(n-1)$ solution $S(\boldsymbol{y}^*)$

Proof.

Let X^* be an optimal solution of $(\mathcal{P}_{\mathrm{R}})$ satisfying $X^* S(\boldsymbol{y}^*) = O$.
From Sylvester's rank inequality,

$$\begin{aligned} \operatorname{rank}(X^*) &\leq n - \operatorname{rank}\{S(\boldsymbol{y}^*)\} + \operatorname{rank}\{X^* S(\boldsymbol{y}^*)\} \\ &= n - \underbrace{\operatorname{rank}\{S(\boldsymbol{y}^*)\}}_{\geq n-1} \leq 1. \quad \square \end{aligned}$$

Recent Research based on Rank

- Extended Trust-region subproblems
  Jeyakumar[2014], Hsia and Sheu[2013], Locatelli[2016]

- Diagonal QCQPs Burer and Ye[2020]
- Exactness by faces of convex lagrangian multipliers
  Wang and Kılınç-Karzan[2021]
- Rank-one generated cones
  Argue, Kılınç-Karzan and Wang[2020]

Idea for Diagonal Matrices $Q^0, \ldots, Q^m$

Assume: $Q^0, \ldots, Q^m$ are diagonal matrices.

$$S(\boldsymbol{y}^*) = \begin{bmatrix} S_{11} & & & \\ & S_{22} & & \\ & & \ddots & \\ & & & S_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

It holds that $\text{rank}\{S(\boldsymbol{y}^*)\} \geq n-1$ if
at least $(n-1)$ of $S_{11}, \ldots, S_{nn}$ are nonzero.

Exactness Condition for Diagonal QCQPs

Non-homogeneous version of QCQP:

$$\begin{aligned} v^* \ := \ \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} + 2(\boldsymbol{q}^0)^{\mathrm{T}} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} + 2(\boldsymbol{q}^p)^{\mathrm{T}} \boldsymbol{x} \le b_p, \quad p \in [m], \end{aligned}$$

Define $S(y) = Q^0 + \sum y_p Q^p, \quad s(y) = q_0 + \sum y_p q_p.$

Theorem [Burer and Ye, 2020]

Suppose $Q^0, Q^1, \dots, Q^m$ are diagonal matrices.
If the following system is infeasible for all $k \in \{1, \dots, n\}$:

$$\boldsymbol{y} \ge 0,\ S(\boldsymbol{y}) \succeq 0,\ S(\boldsymbol{y})_{kk} = 0,\ s(\boldsymbol{y})_k = 0,$$

then $v^* = v^*_{\mathrm{SDP}}$.

Problem and Our Objective

Problem:

- Diagonal property is too strong.
  (c.f., simultaneous diagonalization, adding auxiliary variables)
- Range of applicable problems is still small.

Objective of our research

To expand the range of applicable problems by

- dropping the diagonal property
- employing the sparsity structure of QCQPs instead

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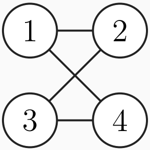
Aggregated Sparsity Pattern of QCQP: $G(\mathcal{V}, \mathcal{E})$

$$\mathcal{V} := [n],$$
$$\mathcal{E} := \left\{ (i,j) \in \mathcal{V}^2 \;\middle|\; i \neq j,\ Q^p_{ij} \neq 0 \text{ for some } p \in \{0, \dots, m\} \right\}.$$



ex. $\min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \right\}$

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Matrix $S(y)$ under Sparsity Structure G

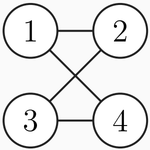
$S(\boldsymbol{y})$ has the same sparsity structure as that of QCQP.

Observation

$$S(\boldsymbol{y})_{ij} = Q^0_{ij} + y_1 Q^1_{ij} + \cdots + y_m Q^m_{ij} = 0 \quad \forall \boldsymbol{y} \in \mathbb{R}^m \quad \forall (i,j) \notin \mathcal{E}$$

ex.

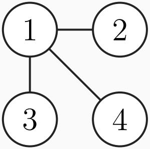
$$S(\boldsymbol{y}) = \begin{bmatrix} -y_1 & -2+y_1 & 0 & 2 \\ -2+y_1 & +4y_1 & -1-y_1 & 0 \\ 0 & -1-y_1 & 5+6y_1 & 1+y_1 \\ 2 & 0 & 1+y_1 & -4-2y_1 \end{bmatrix}$$



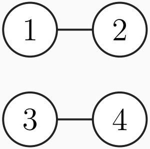
Forest and Bipartite Graph

Let $G(\mathcal{V}, \mathcal{E})$ be a nonempty graph.

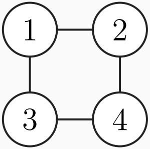
	Cycle		
G	Odd length	Even length	#Components
Tree			1
Forest			≥ 1
Bipartite		allowed	≥ 1



Tree



Forest



Bipartite

Idea for $S(y)$ with Structured Sparsity

$v^* = v^*_{\mathrm{SDP}}$

$\iff$ $(\mathcal{P}_{\mathrm{R}})$ has an optimal X^* satisfying $\mathrm{rank}(X^*) \leq 1$

$\Longleftarrow$ $(\mathcal{D}_R)$ has an optimal $\boldsymbol{y}^*$ satisfying $\mathrm{rank}\{S(\boldsymbol{y}^*)\} \geq n-1$

$\Longleftarrow$ $\mathrm{rank}\{S(\boldsymbol{y})\} \geq n-1 \quad \forall \boldsymbol{y} \geq \mathbf{0}$ satisfying $S(\boldsymbol{y}) \succeq O$

$\Longleftarrow \begin{cases} G\text{: connected bipartite,} \\ S(\boldsymbol{y}) \succeq O,\ S(\boldsymbol{y})_{ij} > 0\ \forall (i,j) \in \mathcal{E} \end{cases}$ [Grone et al., 1992]

If $\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ S(\boldsymbol{y})_{ij} \leq 0$ has no solutions,

we can obtain $S(\boldsymbol{y})_{ij} > 0\ \forall (i,j) \in \mathcal{E}$.

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New Exactness Condition

Exactness Condition for QCQPs with Bipartite Structures

If G is a bipartite and the system

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \tag{1}$$

has no solutions for all $(k, \ell) \in \mathcal{E}$, then $v^* = v^*_{\mathrm{SDP}}$.

- Checking $|\mathcal{E}|$ feasibility systems is required.
- (1) is a tractable problem.

Comparison of Exactness Conditions

	Graph G	Systems to check
Burer & Ye[1]	no edges	$\mathcal{S}_=$ for all (k,ℓ) such that $k=\ell$
Azuma et al.[2]	forest	$\mathcal{S}_=$ for all $(k,\ell) \in \mathcal{E}$
Proposed method	bipartite	$\mathcal{S}_\leq$ for all $(k,\ell) \in \mathcal{E}$

a LP or a SDP: tractable problem

where systems are:

$$\text{find } \ \boldsymbol{y} \geq 0 \ \text{ such that } \ S(\boldsymbol{y}) \succeq O, \ [S(\boldsymbol{y})]_{k\ell} \diamondsuit 0. \qquad (\mathcal{S}_\diamondsuit)$$

[1] Samuel Burer and Yinyu Ye. "Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs". In: *Mathematical Programming* 181.1 (2020), pp. 1–17.

[2] Godai Azuma et al. "Exact SDP Relaxations of Quadratically Constrained Quadratic Programs with Forest Structures". In: *Journal of Global Optimization* 82.2 (2022), pp. 243–262.

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Sign-definite QCQP

Definition of Sign-definite QCQP

- QCQPs with no sparsity
- For all (i, j), the set $T_{ij} := \{Q^0_{ij}, \dots, Q^m_{ij}\}$ is sign-definite

all nonnegative or all nonpositive

Theorem 2[3] [Sojoudi and Lavaei, 2014]

$$\prod_{(i,j)\in\mathcal{C}} \sigma_{ij} = (-1)^{|\mathcal{C}|} \quad \text{for all cycles } \mathcal{C} \text{ in } G \tag{2}$$

where σ_{ij} is called edge sign:

$$\sigma_{ij} = \begin{cases} +1 & (T_{ij} \text{ has all nonnegative}), \\ -1 & (T_{ij} \text{ has all nonpositive}). \end{cases}$$

[3] Somayeh Sojoudi and Javad Lavaei. “Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure”. In: *SIAM Journal on Optimization* 24.4 (2014), pp. 1746–1778.

Sign-definite QCQP

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Theorem 2[3] … 014]

even-length cycle $\iff$ even number of negative σ_{ij}s

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Sign-definite QCQP

Definition of Sign-definite QCQP

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- For all (i, j), the set $T_{ij} := \{Q_{ij}^0, \ldots, Q_{ij}^m\}$ is sign-definite

all nonnegative or all nonpositive

Theorem 2[3] [... 2014]

odd-length cycle $\Longleftrightarrow$ odd number of negative σ_{ij}s

$$\prod_{(i,j)\in\mathcal{C}} \sigma_{ij} = (-1)^{|\mathcal{C}|} \quad \text{for all cycles } \mathcal{C} \text{ in } G \tag{2}$$

where σ_{ij} is called edge sign:

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Some Results on Sign-definite QCQP

Corollary from our exactness condition.

Nonnegative Off-diagonal

$v^* = v^*_{\mathrm{SDP}}$ holds if $G(\mathcal{V}, \mathcal{E})$ is bipartite, and
all the off-diagonal elements of $Q^0, \ldots, Q^m$ are nonnegative, i.e.,

$$\sigma_{ij} = +1 \text{ for all } (i, j) \in \mathcal{E}$$

Moreover, based on the transformation, we have proved:

Comparison

Our proposed condition is weaker than the condition (2).

Essence of Proof (1)

Transform from general QCQPs to sparse QCQPs.

1. Objective function and constraints have the form:

$$\boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} Q^p_{11} & Q^p_{12} & Q^p_{13} & Q^p_{14} \\ Q^p_{21} & Q^p_{22} & Q^p_{23} & Q^p_{24} \\ Q^p_{31} & Q^p_{32} & Q^p_{33} & Q^p_{34} \\ Q^p_{41} & Q^p_{42} & Q^p_{43} & Q^p_{44} \end{bmatrix} \boldsymbol{x}, \quad \forall p,$$

   where $n = 4$, Q^p: symmetric matrices.

2. Assume remove edges $(1, 3)$ and $(2, 4)$ because of $Q^p_{13} < 0, \;\; Q^p_{24} < 0$.

Essence of Proof (2)

3. New variable $z := -x$ is introduced.
4. It can be written as

$$\begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{cccc|cccc} Q^p_{11} & Q^p_{12} & 0 & Q^p_{14} & 0 & 0 & -\frac{1}{2}Q^p_{13} & 0 \\ Q^p_{21} & Q^p_{22} & Q^p_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & Q^p_{32} & Q^p_{33} & Q^p_{34} & -\frac{1}{2}Q^p_{31} & 0 & 0 & 0 \\ Q^p_{41} & 0 & Q^p_{43} & Q^p_{44} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2}Q^p_{13} & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ -\frac{1}{2}Q^p_{31} & 0 & 0 & 0 & & O & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix}, \ \forall p.$$

5. Off-diagonal elements are all nonnegative with some zero elements (bipartite).

The obtained problem satisfies our simple corollary.

Summary

Summary

- QCQPs whose G is bipartite were analyzed.
- New sufficient condition of $v^* = v^*_{\mathrm{SDP}}$ was proposed.
- It was compared with three existing results.

Future works

- Approximated problems of QCQPs with exact SDP relaxation
- Analysis of QCQPs transformed from general problems

More information is available at arXiv:2204.09509,

"Exact SDP relaxations for quadratic programs with bipartite graph structures."

Thank you for your attention!

Backup Slides

Assumption in This Talk

Assumption

(i) Both ($\mathcal{P}_{\mathrm{R}}$) and ($\mathcal{D}_R$) have optimal solutions, and

(ii) At least one of the following two conditions holds:

(a) the feasible region of ($\mathcal{P}_{\mathrm{R}}$) is bounded, or

(b) the set of optimal solutions for ($\mathcal{D}_R$) is bounded.

- strong duality holds: (Kim and Kojima[4])

  $\exists(X^*, \boldsymbol{y}^*)$: solutions of ($\mathcal{P}_{\mathrm{R}}$) and ($\mathcal{D}_R$) such that

$$X^* S(\boldsymbol{y}^*) = O.$$

[4] Sunyoung Kim and Masakazu Kojima. *Strong duality of a conic optimization problem with a single hyperplane and two cone constraints.* arXiv:2111.03251v2. 2021.

Cycle Basis

$\mathcal{C} := \{\mathcal{C}_1, \ldots, \mathcal{C}_\kappa\}$: set of (simple) cycles

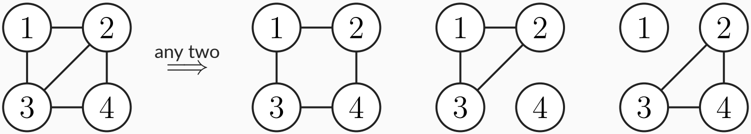
$A \triangle B$: symmetric difference of A and B

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

$\mathcal{C}$ is called a cycle basis of G

$\iff$
- ∘ allows any cycle in G to be expressed by $\triangle$ of its elements
- ∘ be a minimum set

Ex.



Estimation of Rank of Sparse Matrices

Let M be an $n \times n$ symmetric matrix,
$G(\mathcal{V}, \mathcal{E})$ be a graph.

For tree [Johnson et al., 2003, Corollary 3.9]

$$\left.\begin{array}{l} G \text{ is tree} \\ M \succeq O \\ M_{ij} \neq 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n-1$$

For bipartite [Grone et al., 1992, Proposition 1]

$$\left.\begin{array}{l} G \text{ is connected and bipartite} \\ M \succeq O,\ M\mathbf{1} > 0 \\ M_{ij} > 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n-1$$

where $\mathbf{1}$ is the one vector $[1 \ \cdots \ 1]^{\mathrm{T}}$.

Instance for SDP relaxation (QCQP side)

Example 1[5]

$$
\begin{aligned}
v^* \;=\; \min \quad & x^2 + y^2 \\
\text{s.t.} \quad & y^2 \geq 1,\; x^2 - xy \geq 1,\; x^2 + xy \geq 1
\end{aligned}
$$

From last two inequality,

- $$\left.\begin{array}{rcrcr} xy > 0 & \Longrightarrow & xy & > & -xy \\ xy < 0 & \Longrightarrow & -xy & > & xy \end{array}\right\} \Longrightarrow \quad x^2 \geq |x||y| + 1$$
- $$x^2 \geq 1.$$

$$\therefore \quad x^2 + y^2 \geq (|x||y| + 1) + 1 \geq 3$$

[5] Luo-Lecture14.

Instance for SDP relaxation (SDP side)

$$
\begin{aligned}
v^* &= \min \left\{ x^2 + y^2 \,\middle|\, y^2 \geq 1,\ x^2 - xy \geq 1,\ x^2 + xy \geq 1 \right\} \\
&= \min \left\{ \boldsymbol{x}^{\mathrm{T}} I \boldsymbol{x} \,\middle|\, \begin{array}{l} \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} \geq 1, \\ \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \boldsymbol{x} \geq 1,\ \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} \boldsymbol{x} \geq 1 \end{array} \right\} \\
&\geq \min \left\{ I \bullet X \,\middle|\, \begin{array}{l} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet X \geq 1,\ X \succeq O \\ \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \bullet X \geq 1,\ \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} \bullet X \geq 1 \end{array} \right\} \\
&= \min \left\{ X_{11} + X_{22} \,\middle|\, X_{22} \geq 1,\ X_{11} - X_{12} \geq 1,\ X_{11} + X_{12} \geq 1,\ X \succeq O \right\}
\end{aligned}
$$

$X = I$ is feasible $\quad \Longrightarrow \quad v^*_{\mathrm{SDP}} \leq 2 < 3 \leq v^*$

QCQP's Applications

Used in various problems:

- MAX-CUT, MAX-CLIQUE
- sensor (facility) location problem, pooling problem
- optimal flow problem, polynomial optimization
- (robust / sparse) principal component analysis, phase retrieval

Alternative Result for Theorem 3.7

Theorem 3.7'

Suppose Assumption 3.8 holds & G is a forest and connected graph.
Then, $v^* = v^*_{\mathrm{SDP}}$ if for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \qquad \textbf{(??)}$$

Ex. $n = 2, \quad m = 1,$

$$\min . \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} \boldsymbol{x} \qquad \text{s.t.} \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \boldsymbol{x} \leq 1.$$

- $\mathcal{E} = \{(1,2),(2,1)\}$
- Systems — only for $(k,\ell) = (1,2)$

$$y_1 \geq 0, \quad \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} + y_1 \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \succeq O, \quad -1 + 4y_1 \leq 0$$

- $S(y) \succeq O \iff y_1 \geq 3 + \frac{3\sqrt{6}}{2} \simeq 6.67$

second inequality $\implies -1 + 4y_1 > 0 \qquad \implies v^* = v^*_{\mathrm{SDP}}$

No solutions

Example 2

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^2 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^3 \boldsymbol{x} \leq 5 \end{aligned}$$

where

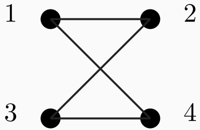
$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \; Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

Sparsity of Example 2

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \; Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$



$$\mathcal{E} = \left\{ \begin{array}{l} (1,2),(2,1),(1,4),(4,1), \\ (2,3),(3,2),(3,4),(4,3) \end{array} \right\}.$$

Systems of Example 2

Consider the problem for $(k, \ell) \in \mathcal{E}$:

$$\begin{aligned} \mu^* = \min \quad & S(\boldsymbol{y})_{k\ell} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \mathbf{0},\ S(\boldsymbol{y}) \succeq O. \end{aligned} \tag{3}$$

(k, ℓ)	$(1, 2)$	$(2, 3)$	$(1, 4)$	$(3, 4)$
μ^*	18.58	12.84	8.897	0.3215

All positives $\implies$ the following systems have no solutions:

$$\boldsymbol{y} \geq \mathbf{0},\ S(\boldsymbol{y}) \succeq O,\ S(\boldsymbol{y})_{k\ell} \leq 0. \tag{1}$$

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}$}$$

$$\downarrow$$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P \right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}^\varepsilon$}$$

- ($\mathcal{P}^\varepsilon$) converges to ($\mathcal{P}$) as $\varepsilon \downarrow 0$.
- If $P \preceq O$, then $v_\varepsilon^* \leq v^*$.

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}$}$$

$$\downarrow$$

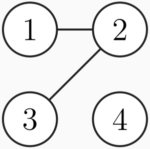
$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \tag{$\mathcal{P}^\varepsilon$}$$

Preferred edges can be added to G

- ($\mathcal{P}^\varepsilon$) converges to ($\mathcal{P}$) as $\varepsilon \downarrow 0$.
- If $P \preceq O$, then $v_\varepsilon^* \leq v^*$.

Instance of Adding Edges

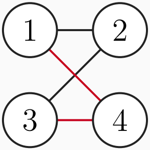
$$\min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 \qquad \right) \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \right\}$$



$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

Instance of Adding Edges

$$\min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P \right) \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \right\}$$



$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Perturbation and Exactness

$\{\varepsilon_t\}_{t=1}^{\infty} \subseteq \mathbb{R}_+$: monotonically decreasing and $\lim_{t \to \infty} \varepsilon_t = 0$

$P \neq O$: $n \times n$ negative semidefinite matrix

Lemma 4.1, Lemma 4.4, and Lemma 4.5

Suppose Assumption 3.6 holds, or
Assumption 3.8 and an additional one hold.

SDP relaxation of $(\mathcal{P}^{\varepsilon})$ is exact for all $\varepsilon = \varepsilon_1, \varepsilon_2, \ldots$ $\implies$ $v^* = v^*_{\mathrm{SDP}}$.

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of ($\mathcal{P}$).

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest and connected.
Then, $v^* = v^*_{\text{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \quad \textbf{(??)}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite and connected.
Then, $v^* = v^*_{\text{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \quad (1)$$

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of ($\mathcal{P}$).

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \tag{??}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \tag{1}$$

Sketch of Proof

Proof of Theorem 4.6.

1. Let $\mathcal{F}$ be the set of additional edges.
2. Define $P \preceq O$ as

$$P_{ij} = \begin{cases} -\deg(i) & \text{if } i = j, \\ 1 & \text{if } (i,j) \in \mathcal{F} \text{ or } (j,i) \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

3. ($\mathcal{P}^\varepsilon$) satisfies assumptions of Theorem 3.10.
   $\implies$ SDP relaxation of ($\mathcal{P}^\varepsilon$) is exact.
4. Using Lemma 4.4 (4.5), we conclude $v^* = v^*_{\mathrm{SDP}}$. □

Previous Works

- Trust-region subproblems (TRS: QCQP with one constraint)
  Yakubovich[1971]
- Extended TRS (TRS + linear constraints)
  Jeyakumar[2014], Hsia and Sheu[2013], Locatelli[2016]
- QCQPs with sign-definiteness
  Kim and Kojima[2003], Sojoudi and Lavaei[2014]

- Exactness by faces of convex lagrangian multipliers
  Wang and Kılınç-Karzan[2021]
- Rank-one generated cones
  Argue, Kılınç-Karzan and Wang[2020]

Proposed Condition Covers Condition (2)

Proposition

If a given $(\mathcal{P})$ satisfies the condition (2),
proposed condition can detect the exactness of its SDP relaxation.

Idea:

We develop conversion method of QCQPs such that

- The obtained QCQP has bipartite sparsity.
- The obtained QCQP satisfies proposed condition:
  $\forall (k, \ell) \in \mathcal{E}$, the system (1) has no solutions.