Tight Semidefinite Relaxations for Sign-indefinite QCQPs with Bipartite Structures

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QCQP: Quadratically Constrained Quadratic Programming

Consider a quadratic programming with quadratic constraints:

$$
\begin{aligned}
v^* \;:=\; \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p, \quad p \in [m] := \{1,\ldots,m\},
\end{aligned}
\tag{$\mathcal{P}$}
$$

- Generally, non-convex & NP-hard
- Semidefinite programming (SDP) relaxation

Applications

Binary programming, MAX-CUT, optimal flow problems,...

Semidefinite Programming (SDP) Relaxation

$$
\begin{aligned}
v^* &= \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \,\middle|\, \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p \quad \forall p \in [m] \right\} \\
&= \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} \\
&\ge \min \left\{ Q^0 \bullet X \,\middle|\, \begin{array}{l} X \succeq \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} =: v^*_{\mathrm{SDP}} \qquad (\mathcal{P}_{\mathrm{R}})
\end{aligned}
$$

where

- $Q^p \bullet X := \sum_{i,j} Q^p_{ij} X_{ij}$,
- $X \succeq O \iff X$ is positive semidefinite.

Pros: calculatable in polynomial time.

Cons: $v^* \neq v^*_{\mathrm{SDP}}$ in general

Tightness for SDP Relaxation

The following equality holds:

$$v^* = \min\left\{ Q^0 \bullet X \;\middle|\; \begin{array}{l} X \succeq O \\ Q^p \bullet X \le b_p \quad \forall p \in [m] \end{array} \right\} = v^*_{\mathrm{SDP}}$$

$\iff$ rank-1 solution X^* exists

Tight SDP relaxation $\implies$

- Original QCQP is exactly solvable (in theorically)
- The gap between a class of QCQPs and their relaxations is identified.

Motivation

What conditions of QCQPs guarantee the tightness?

Assumption in This Talk

Assumption

(i) Both ($\mathcal{P}_{\mathrm{R}}$) and ($\mathcal{D}_{R}$) have optimal solutions, and

(ii) At least one of the following two conditions holds:

(a) the feasible region of ($\mathcal{P}_{\mathrm{R}}$) is bounded, or

(b) the set of optimal solutions for ($\mathcal{D}_{R}$) is bounded.

- strong duality holds: (Kim and Kojima[1])

  $\exists(X^*, \boldsymbol{y}^*)$: solutions of ($\mathcal{P}_{\mathrm{R}}$) and ($\mathcal{D}_{R}$) such that

$$X^* S(\boldsymbol{y}^*) = O.$$

[1] Sunyoung Kim and Masakazu Kojima. *Strong duality of a conic optimization problem with a single hyperplane and two cone constraints.* arXiv:2111.03251v2. 2021.

Outline

1. Introduction
2. Tightness for sign-definite QCQPs
3. Extension to sign-indefinite QCQPs
   - Tightness & solution's rank of dual SDP relaxation
   - Upper bound for rank of matrices
   - Tightness conditions under bipartite structure
   - Adaptation for disconnected structures
4. Examples
5. Summary

Sign-definite QCQP

Definition

Sign-definite QCQP when

$$\text{same index} \implies \text{same sign } (\geq \text{ or } \leq) \quad \text{among } Q^0, \ldots, Q^m$$

Ex.:

$$\min.\ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \quad \text{s.t. } \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq 10,\ p \in [3]$$

$$Q^0 := \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 := \begin{bmatrix} 5 & -2 & 0 & 1 \\ -2 & -1 & -3 & 0 \\ 0 & -3 & 3 & 1 \\ 1 & 0 & 1 & 4 \end{bmatrix},$$

$$Q^2 := \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad Q^3 := \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

QCQP's Sparsity

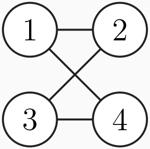
Aggregated Sparsity Pattern Graph $G(V, E)$

= Nonzero pattern of variable matrices in dual problem

$$V := \{1, \ldots, n\}, \quad E := \{(i, j) \mid i \neq j,\ [Q^p]_{ij} \neq 0 \text{ for some } Q^p\}.$$

Ex.:

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$



Edge sign:

$$\sigma_{ij} = \begin{cases} -1 & \text{if } (i, j)\text{th elements} \leq 0 \\ +1 & \text{if } (i, j)\text{th elements} \geq 0. \end{cases}$$

Ex. $\sigma_{12} = \sigma_{23} = -1, \quad \sigma_{14} = \sigma_{34} = +1.$

Tightness of Sign-definite QCQP

Cycle-based conditions:

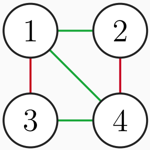
Theorem 2[2] **[Sojoudi and Lavaei, 2014]**

Tight if the following equation holds

$$\prod_{(i,j)\in\mathcal{C}} \sigma_{ij} = (-1)^{|\mathcal{C}|} \quad \text{for any cycles } \mathcal{C} \text{ in } G \tag{1}$$

i.e., even-length cycle (odd) $\iff$ the number of negative σ_{ij} are even (odd)

Ex. $\sigma_i j = -1$, $\sigma_i j = +1$



[2] Somayeh Sojoudi and Javad Lavaei. "Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure". In: *SIAM Journal on Optimization* 24.4 (2014), pp. 1746–1778.

Problem and Our Objective

Problem:

- Only a few problems are sign-definite QCQPs.

Objective of our research

To expand the range of applicable problems by

- dropping the sign-definite condition
- employing the rank of dual SDP relaxation instead

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Primal/Dual Problems of SDP Relaxation

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \,\middle|\, \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p,\ p \in [m] \right\} \qquad (\mathcal{P})$$

$\swarrow$

Primal

$$\begin{array}{rl} \min\limits_{X} & Q^0 \bullet X \\ \text{s.t.} & Q^p \bullet X \leq b_p,\ p \in [m] \\ & X \succeq O \end{array} \qquad (\mathcal{P}_R)$$

Dual

$$\begin{array}{rl} \max\limits_{\boldsymbol{y}\in\mathbb{R}^m} & -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \geq \boldsymbol{0}, \\ & S(\boldsymbol{y}) \succeq O \end{array} \qquad (\mathcal{D}_R)$$

Variable on dual side: $(\boldsymbol{y}, S(\boldsymbol{y}))$ where

$$S(\boldsymbol{y}) := Q^0 + \sum_{p=1}^{m} y_p Q^p \quad \text{for } \boldsymbol{y} \in \mathbb{R}^m.$$

Dual Solution of rank-$(n-1)$ is Important for Tightness

Primal

$$\begin{aligned} \min_{X} \quad & Q^0 \bullet X \\ \text{s.t.} \quad & Q^p \bullet X \leq b_p,\ p \in [m] \\ & X \succeq O \end{aligned} \qquad (\mathcal{P}_R)$$

Dual

$$\begin{aligned} \max_{\boldsymbol{y}\in\mathbb{R}^m} \quad & -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \boldsymbol{0}, \\ & S(\boldsymbol{y}) \succeq O \end{aligned} \qquad (\mathcal{D}_R)$$

Tight if

Rank-1 solution X^*

Dual Solution of rank-$(n-1)$ is Important for Tightness

Primal

$$\begin{aligned} \min_{X} \quad & Q^0 \bullet X \\ \text{s.t.} \quad & Q^p \bullet X \leq b_p, \; p \in [m] \qquad (\mathcal{P}_R) \\ & X \succeq O \end{aligned}$$

Dual

$$\begin{aligned} \max_{\boldsymbol{y} \in \mathbb{R}^m} \quad & -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \boldsymbol{0}, \qquad (\mathcal{D}_R) \\ & S(\boldsymbol{y}) \succeq O \end{aligned}$$

Tight if **Rank-1 solution X^*** $\Longleftarrow$ **Rank-$(n-1)$ solution $S(\boldsymbol{y}^*)$**

under strong duality

Proof. There exists X^* satisfying $X^* S(\boldsymbol{y}^*) = O$.
From Sylvester's rank inequality,

$$\begin{aligned} \operatorname{rank}(X^*) &\leq n - \operatorname{rank}\{S(\boldsymbol{y}^*)\} + \operatorname{rank}\{X^* S(\boldsymbol{y}^*)\} \\ &= n - \underbrace{\operatorname{rank}\{S(\boldsymbol{y}^*)\}}_{\geq n-1} \leq 1. \quad \square \end{aligned}$$

Recent Tightness Conditions

Based on dual SDP or its rank:

- Extended Trust-region subproblems
  Jeyakumar[2014], Hsia and Sheu[2013], Locatelli[2016]
- Diagonal QCQPs   Burer and Ye[2020]
- Tightness by faces of convex lagrangian multipliers
  Wang and Kılınç-Karzan[2021]
- Rank-one generated cones
  Argue, Kılınç-Karzan and Wang[2020]

8:00 AM - 9:00 AM	IP4 Exactness in Semidefinite Program Relaxations and Its Implications Fatma Kilinc-Karzan, *Carnegie Mellon University, U.S.*	Grand Ballroom B/C/D, 2nd floor

Sparsity Pattern of $S(\boldsymbol{y})$ under G

$S(\boldsymbol{y})$ has the same sparsity structure as that of QCQP.

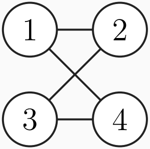
Observation

$$[S(\boldsymbol{y})]_{ij} = [Q^0]_{ij} + \sum_{p \in [m]} y_p [Q^p]_{ij} = 0 \quad \forall \boldsymbol{y} \in \mathbb{R}^m \quad \forall (i,j) \notin E$$

Ex.:

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

$$S(\boldsymbol{y}) = \begin{bmatrix} -y_1 & -2 - y_1 & 0 & 2 \\ -2 - y_1 & +4y_1 & -1 - y_1 & 0 \\ 0 & -1 - y_1 & 5 + 6y_1 & 1 + y_1 \\ 2 & 0 & 1 + y_1 & -4 - 2y_1 \end{bmatrix}$$



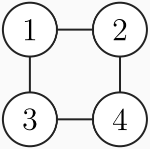
Estimation of Rank of Structured Matrices

$\mathbf{1}$ is the one vector $[1 \ \cdots \ 1]^{\mathrm{T}}$.

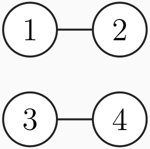
Proposition 1 of [Grone et al., 1992]

$$\left.\begin{array}{l} G \text{ is connected and bipartite} \\ M \succeq O, \ M\mathbf{1} > 0 \\ M_{ij} > 0 \quad \forall (i,j) \in \boldsymbol{E} \end{array}\right\} \implies \operatorname{rank} M \geq n-1$$

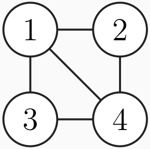
Bipartite graph if G has no odd-length cycles.



✓ Bipartite



NG: disconnected



NG: odd-length cycles

Idea for $S(y)$ with Structured Sparsity

Assume G is bipartite and connected.

To show the tightness, it suffices that ...

at least one optimal $\boldsymbol{y}^*$ satisfies $\operatorname{rank}\{S(\boldsymbol{y}^*)\} \geq n-1$

$\Longleftarrow$ all optimal solutions $\boldsymbol{y}^*$ satisfies $\operatorname{rank}\{S(\boldsymbol{y}^*)\} \geq n-1$

$\Longleftarrow$ any feasible point $\boldsymbol{y}$ satisfies $\operatorname{rank}\{S(\boldsymbol{y})\} \geq n-1$

$\Longleftarrow$ any feasible point $\boldsymbol{y}$ satisfies $S(\boldsymbol{y}) \succeq O$, $[S(\boldsymbol{y})]_{ij} > 0 \; \forall (i,j) \in E$

$\Longleftarrow$ the following equation has no solutions for any $(i,j) \in E$:

$$\boldsymbol{y} \geq 0, \quad S(\boldsymbol{y}) \succeq O, \quad [S(\boldsymbol{y})]_{ij} \leq 0.$$

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$$\boldsymbol{y} \geq 0, \quad S(\boldsymbol{y}) \succeq O, \quad [S(\boldsymbol{y})]_{ij} \leq 0.$$

Our Proposed Tightness Condition

Tightness Condition for QCQPs with Bipartite Structures

Tight if G is a bipartite and connected graph and the system

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \tag{2}$$

has no solutions for any $(k, \ell) \in E$.

- $|E|$ feasibility systems to check
- Structure-based condition
- no dependence on sign-definiteness

Bad: connectivity of G is required.

$\Longrightarrow$ removed by perturbation (next slide)

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P \right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P}^\varepsilon)$$

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P}^\varepsilon)$$

$\downarrow$ converging as $\varepsilon \downarrow 0$

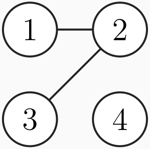
$$v^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P})$$

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P}^\varepsilon)$$

Ex. $\min \left\{ \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10 \right\}$



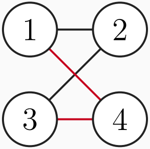
$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

ε-Perturbed QCQPs

Let $P \neq O \in \mathbb{S}^n$ and $\varepsilon > 0$

$$v_\varepsilon^* = \min \left\{ \boldsymbol{x}^\mathrm{T} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \mid \boldsymbol{x}^\mathrm{T} Q^p \boldsymbol{x} \leq b_p \quad \forall p \in [m] \right\} \qquad (\mathcal{P}^\varepsilon)$$

Ex. $\min \left\{ \boldsymbol{x}^\mathrm{T} \left(Q^0 + \varepsilon P\right) \boldsymbol{x} \mid \boldsymbol{x}^\mathrm{T} Q^1 \boldsymbol{x} \leq 10 \right\}$



$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Perturbation and Tightness

Theoretical background of ε-perturbed QCQPs.

Lemma

The original QCQP admits tight relaxation if there exists

$$\{\varepsilon_t\}_{t=1}^{\infty} \subseteq \mathbb{R}_+ : \text{monotonically decreasing and } \lim_{t\to\infty} \varepsilon_t = 0$$

$$P \neq O : n \times n \text{ negative semidefinite matrix}$$

such that SDP relaxation of $(\mathcal{P}^{\varepsilon_t})$ is tight for any ε_t.

Comparison of Tightness Conditions

	G	Systems to check
Burer & Ye[3]	no edges	$\mathcal{S}_=$ for all (k,ℓ) such that $k=\ell$
Azuma et al.[4]	forest	$\mathcal{S}_=$ for all $(k,\ell) \in \mathcal{E}$
Proposed method	bipartite	$\mathcal{S}_\leq$ for all $(k,\ell) \in \mathcal{E}$

a LP or a SDP: tractable problem

where systems are:

$$\text{find } \boldsymbol{y} \geq 0 \text{ such that } S(\boldsymbol{y}) \succeq O, \ [S(\boldsymbol{y})]_{k\ell} \diamondsuit 0. \qquad (\mathcal{S}_\diamondsuit)$$

[3] Samuel Burer and Yinyu Ye. "Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs". In: *Mathematical Programming* 181.1 (2020), pp. 1–17.

[4] Godai Azuma et al. "Exact SDP Relaxations of Quadratically Constrained Quadratic Programs with Forest Structures". In: *Journal of Global Optimization* 82.2 (2022), pp. 243–262.

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Example 1

Ex. $n = 2, \quad m = 1,$

$$\min . \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} \boldsymbol{x} \qquad \text{s.t.} \quad \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \boldsymbol{x} \le 1.$$

- $\mathcal{E} = \{(1,2), (2,1)\}$
- Systems — only for $(k, \ell) = (1, 2)$

$$y_1 \ge 0, \quad \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} + y_1 \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \succeq O, \quad -1 + 4y_1 \le 0$$

- $S(y) \succeq O \iff y_1 \ge 3 + \frac{3\sqrt{6}}{2} \simeq 6.67$

second inequality $\implies -1 + 4y_1 > 0 \qquad \implies v^* = v^*_{\mathrm{SDP}}$

No solutions

Summary

Summary

- QCQPs without sign-definiteness were analyzed.
- New sufficient condition of tightness was proposed.
- It was compared with three existing results.

Future works

- How to expand applicable problems from bipartite structures.
- How to transform more general problems to QCQP admitting tight SDP relaxation.

More information is available at DOI:10.1007/s10898-022-01268-3,
“Exact SDP relaxations for quadratic programs with bipartite graph structures.”

Thank you for your attention!

Backup Slides

Essence of Proof (1)

Transform from general QCQPs to sparse QCQPs.

1. Objective function and constraints have the form:

$$\boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} Q^p_{11} & Q^p_{12} & \color{red}{Q^p_{13}} & Q^p_{14} \\ Q^p_{21} & Q^p_{22} & Q^p_{23} & \color{red}{Q^p_{24}} \\ \color{red}{Q^p_{31}} & Q^p_{32} & Q^p_{33} & Q^p_{34} \\ Q^p_{41} & \color{red}{Q^p_{42}} & Q^p_{43} & Q^p_{44} \end{bmatrix} \boldsymbol{x}, \quad \forall p,$$

   where $n = 4$, Q^p: symmetric matrices.

2. Assume remove edges $(1, 3)$ and $(2, 4)$ because of $Q^p_{13} < 0, \;\; Q^p_{24} < 0$.

Essence of Proof (2)

3. New variable $z := -x$ is introduced.
4. It can be written as

$$\begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{cccc|cccc} Q^p_{11} & Q^p_{12} & 0 & Q^p_{14} & 0 & 0 & -\frac{1}{2}Q^p_{13} & 0 \\ Q^p_{21} & Q^p_{22} & Q^p_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & Q^p_{32} & Q^p_{33} & Q^p_{34} & -\frac{1}{2}Q^p_{31} & 0 & 0 & 0 \\ Q^p_{41} & 0 & Q^p_{43} & Q^p_{44} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2}Q^p_{13} & 0 & & & & \\ 0 & 0 & 0 & 0 & & & O & \\ -\frac{1}{2}Q^p_{31} & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix}, \ \forall p.$$

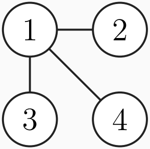
5. Off-diagonal elements are all nonnegative
   with some zero elements (bipartite).

The obtained problem satisfies our simple corollary.

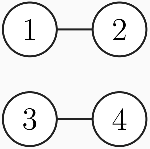
Forest and Bipartite Graph

Let $G(\mathcal{V}, \mathcal{E})$ be a nonempty graph.

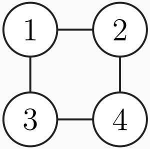
	Cycle		
G	Odd length	Even length	#Components
Tree			1
Forest			≥ 1
Bipartite		allowed	≥ 1



Tree



Forest



Bipartite

Cycle Basis

$\mathcal{C} := \{\mathcal{C}_1, \ldots, \mathcal{C}_\kappa\}$: set of (simple) cycles

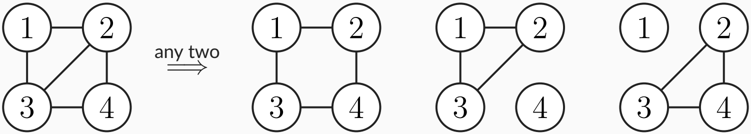
$A \triangle B$: symmetric difference of A and B

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

$\mathcal{C}$ is called a cycle basis of G

$\iff$
- ∘ allows any cycle in G to be expressed by $\triangle$ of its elements
- ∘ be a minimum set

Ex.



Estimation of Rank of Sparse Matrices

Let M be an $n \times n$ symmetric matrix,
$G(\mathcal{V}, \mathcal{E})$ be a graph.

For tree **[Johnson et al., 2003, Corollary 3.9]**

$$\left.\begin{array}{l} G \text{ is tree} \\ M \succeq O \\ M_{ij} \neq 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n - 1$$

For bipartite **[Grone et al., 1992, Proposition 1]**

$$\left.\begin{array}{l} G \text{ is connected and bipartite} \\ M \succeq O,\ M\mathbf{1} > 0 \\ M_{ij} > 0 \quad \forall (i,j) \in \mathcal{E}_M \end{array}\right\} \implies \operatorname{rank} M \geq n - 1$$

where $\mathbf{1}$ is the one vector $[1 \ \cdots \ 1]^{\mathrm{T}}$.

Instance for SDP relaxation (QCQP side)

Example 1[5]

$$\begin{aligned} v^* \;=\; \min \quad & x^2 + y^2 \\ \text{s.t.} \quad & y^2 \geq 1,\; x^2 - xy \geq 1,\; x^2 + xy \geq 1 \end{aligned}$$

From last two inequality,

- $$\left.\begin{array}{lcrcr} xy > 0 & \Longrightarrow & xy & > & -xy \\ xy < 0 & \Longrightarrow & -xy & > & xy \end{array}\right\} \Longrightarrow \quad x^2 \geq |x||y| + 1$$
- $$x^2 \geq 1.$$

$$\therefore \quad x^2 + y^2 \geq (|x||y| + 1) + 1 \geq 3$$

[5] Luo-Lecture14.

Instance for SDP relaxation (SDP side)

$$
\begin{aligned}
v^* &= \min \left\{ x^2 + y^2 \,\middle|\, y^2 \ge 1,\ x^2 - xy \ge 1,\ x^2 + xy \ge 1 \right\} \\
&= \min \left\{ \boldsymbol{x}^{\mathrm{T}} I \boldsymbol{x} \,\middle|\, \begin{array}{l} \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} \ge 1, \\ \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \boldsymbol{x} \ge 1,\ \boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} \boldsymbol{x} \ge 1 \end{array} \right\} \\
&\ge \min \left\{ I \bullet X \,\middle|\, \begin{array}{l} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet X \ge 1,\ X \succeq O \\ \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \bullet X \ge 1,\ \begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} \bullet X \ge 1 \end{array} \right\} \\
&= \min \left\{ X_{11} + X_{22} \,\middle|\, X_{22} \ge 1,\ X_{11} - X_{12} \ge 1,\ X_{11} + X_{12} \ge 1,\ X \succeq O \right\}
\end{aligned}
$$

$X = I$ is feasible $\quad \Longrightarrow \quad v^*_{\mathrm{SDP}} \le 2 < 3 \le v^*$

QCQP's Applications

Used in various problems:

- MAX-CUT, MAX-CLIQUE
- sensor (facility) location problem, pooling problem
- optimal flow problem, polynomial optimization
- (robust / sparse) principal component analysis, phase retrieval

Alternative Result for Theorem 3.7

Theorem 3.7'

Suppose Assumption 3.8 holds & G is a forest and connected graph.
Then, $v^* = v^*_{\mathrm{SDP}}$ if for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \quad \textbf{(??)}$$

Example 2

$$
\begin{aligned}
\min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \le 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^2 \boldsymbol{x} \le 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^3 \boldsymbol{x} \le 5
\end{aligned}
$$

where

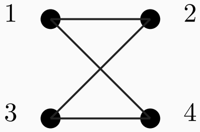
$$
Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \; Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},
$$

$$
Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.
$$

Sparsity of Example 2

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$



$$\mathcal{E} = \left\{ \begin{array}{l} (1,2),(2,1),(1,4),(4,1), \\ (2,3),(3,2),(3,4),(4,3) \end{array} \right\}.$$

Systems of Example 2

Consider the problem for $(k, \ell) \in \mathcal{E}$:

$$\begin{aligned} \mu^* = \min \quad & S(\boldsymbol{y})_{k\ell} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \mathbf{0},\ S(\boldsymbol{y}) \succeq O. \end{aligned} \tag{3}$$

(k, ℓ)	$(1, 2)$	$(2, 3)$	$(1, 4)$	$(3, 4)$
μ^*	18.58	12.84	8.897	0.3215

All positives $\implies$ the following systems have no solutions:

$$\boldsymbol{y} \geq \mathbf{0},\ S(\boldsymbol{y}) \succeq O,\ S(\boldsymbol{y})_{k\ell} \leq 0. \tag{2}$$

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of ($\mathcal{P}$).

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest <u>and connected</u>.
Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0, \ S(\boldsymbol{y}) \succeq O, \ [S(\boldsymbol{y})]_{k\ell} = 0, \qquad \textbf{(??)}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite <u>and connected</u>.
Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0, \ S(\boldsymbol{y}) \succeq O, \ [S(\boldsymbol{y})]_{k\ell} \leq 0, \qquad (2)$$

Result for Disconnected Sparsity Structures

Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of $(\mathcal{P})$.

Theorem 4.2

Suppose Assumption 3.6 holds & G is a forest.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \quad \textbf{(??)}$$

Theorem 4.6

Suppose Assumption 4.3 holds & G is a bipartite.

Then, $v^* = v^*_{\mathrm{SDP}}$ if, for all $(k, \ell) \in \mathcal{E}$, the following system has no solutions:

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} \leq 0, \quad (2)$$

Sketch of Proof

Proof of Theorem 4.6.

1. Let $\mathcal{F}$ be the set of additional edges.
2. Define $P \preceq O$ as

$$P_{ij} = \begin{cases} -\deg(i) & \text{if } i = j, \\ 1 & \text{if } (i,j) \in \mathcal{F} \text{ or } (j,i) \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

3. ($\mathcal{P}^{\varepsilon}$) satisfies assumptions of Theorem 3.10.
   $\implies$ SDP relaxation of ($\mathcal{P}^{\varepsilon}$) is tight.
4. Using Lemma 4.4 (4.5), we conclude $v^* = v^*_{\mathrm{SDP}}$. □

Previous Works

- Trust-region subproblems (TRS: QCQP with one constraint)
  Yakubovich[1971]
- Extended TRS (TRS + linear constraints)
  Jeyakumar[2014], Hsia and Sheu[2013], Locatelli[2016]
- QCQPs with sign-definiteness
  Kim and Kojima[2003], Sojoudi and Lavaei[2014]

- Tightness by faces of convex lagrangian multipliers
  Wang and Kılınç-Karzan[2021]
- Rank-one generated cones
  Argue, Kılınç-Karzan and Wang[2020]

Proposed Condition Covers Condition (1)

Proposition

If a given ($\mathcal{P}$) satisfies the condition (1),
proposed condition can detect the tightness of its SDP relaxation.

Idea:

We develop conversion method of QCQPs such that

- The obtained QCQP has bipartite sparsity.
- The obtained QCQP satisfies proposed condition:
  $\forall(k, \ell) \in \mathcal{E}$, the system (2) has no solutions.