High-rank Solution of Sum-of-Squares Relaxations for Exact Matrix Completion

Godai Azuma [1]   Sunyoung Kim [2]   Makoto Yamashita [3]
[1]Aoyama Gakuin University   [2]Ewha Womans University   [3]Tokyo Institute of Technology

ISMP 2024, FB36 (July 26, 2024)

This work was supported by   JSPS KAKENHI 24K20738, JP22KJ1307 and JP20H04145
NRF 2021-R1A2C1003810

Target Problems

Consider a problem in which a matrix completion problem is hidden.

$$\begin{aligned} \min_{\boldsymbol{z}\in\mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & \boldsymbol{h}(\boldsymbol{z}) = 0. \end{aligned} \tag{P}$$

- Polynomial optimization problem (POP) of $\boldsymbol{z}$
- $\boldsymbol{h}(\boldsymbol{z})$ is a vector of polynomials of $\boldsymbol{z}$:

$$\boldsymbol{h}(\boldsymbol{z}) = [z_i z_j - A_{ij}]_{(i,j)\in\Lambda}, \quad A_{ij} \text{ is constant,}$$

$$\boldsymbol{u}_2(\boldsymbol{z}) = [1, z_1, \ldots, z_n, z_1^2, z_1 z_2, \ldots, z_n^2]^{\mathrm{T}} \in \mathbb{R}^N := \mathbb{R}^{\binom{n+2}{2}}.$$

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$$\begin{aligned} \min_{\boldsymbol{z}\in\mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & C\boldsymbol{h}(\boldsymbol{z}) = 0. \end{aligned} \qquad (P)$$

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- For $C = I$, the POP formulation for the rank-one matrix completion

Relaxations and Tightness

(P): NP-hard

$\swarrow$ $\searrow$

Level-1 Lasserre's Relaxation	$\Leftarrow$ dual $\Rightarrow$	Level-1 SOS Relaxation
Level-2 Lasserre's Relaxation	$\Leftarrow$ dual $\Rightarrow$	Level-2 SOS Relaxation
$\vdots$		$\vdots$

Theorem 1[1]

If $Q = I_N$ and $C = I_K$, then level-2 Lasserre's relaxation is tight.

(Optimal values of it and (P) coincide)

[1] A. Cosse, L. Demanet. "Stable Rank-One Matrix Completion is Solved by the Level 2 Lasserre Relaxation", Foundations of Computational Mathematics, 21, 891–940, 2021.

Motivation

Question

- Tightness of relaxations hold without $Q = I_N$ and $C = I_K$?
- If so, how large class of problems?

$\Longrightarrow$ Classify exactly solvable problems in nonconvex problems

This talk

- High-rank Q preserving tightness
- In particular, $C = I_K$

Sparsity structure Q will be used to show high-rank

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2. POP Formulation of Matrix Completion
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Rank-one Matrix Completion Problem

Recovering a rank-1 matrix from partially given elements.

Given matrix | Completed matrix

$$A = \begin{bmatrix} 7 & 3 & ? \\ -35 & ? & 10 \\ ? & 9 & ? \end{bmatrix} \longrightarrow \begin{bmatrix} 7 & 3 & -2 \\ -35 & -15 & 10 \\ 21 & 9 & -6 \end{bmatrix} \quad \text{: rank-1}$$

Intuitive formulation:

$$\begin{aligned} \text{find} \quad & X \in \mathbb{R}^{n \times m} \\ \text{s.t.} \quad & \operatorname{rank}(X) = 1, \\ & X_{ij} - A_{ij} = 0, \quad (i,j) \in \overline{\Lambda}. \end{aligned}$$

($\overline{\Lambda}$: index set of given elem)

A_{ij}: given (i,j)th element

Example: $\overline{\Lambda} = \{(1,1),(1,2),(2,1),(2,3),(3,2)\}$

Representation of Rank-one Matrix

To make polynomial optimization (POP),

- introduce $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$
- replace X by $\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$

Then, $$X_{ij} - A_{ij} = 0 \quad \Longleftrightarrow \quad \underbrace{x_i y_j - A_{ij} = 0}_{\text{element of } \boldsymbol{h}(\boldsymbol{z})}, \qquad \forall (i,j) \in \overline{\Lambda}.$$

$$\boxed{\text{POP}} \quad \begin{aligned} \text{find} \quad & \boldsymbol{z} := [x_2, \dots, x_n, \boldsymbol{y}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+m-1} \\ \text{s.t.} \quad & x_1 = 1, \\ & x_i y_j - A_{ij} = 0, \quad (i,j) \in \overline{\Lambda}. \end{aligned}$$

POP for Rank-one Matrix Completion

Finally, we have

$$\begin{aligned} \min_{\boldsymbol{z} \in \mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & \boldsymbol{h}(\boldsymbol{z}) = 0. \end{aligned} \qquad (P)$$

- Introduce $Q \in \mathbb{S}^N$ on objective function.
- Focus
  - not on solving the rank-one matrix completion,
  - but on the tightness for problems in which it is hidden.

Assumption Solution exists and it is unique.

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Assumption: Solution exists and it is unique.

Example of using C

$$\begin{aligned} \min_{\boldsymbol{z}\in\mathbb{R}^5} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & y_1 - \tfrac{3}{2}y_2 - \tfrac{5}{2} = 0, \qquad 5y_2 - 2x_3y_2 + 3 = 0, \\ & 10y_2 + x_2y_1 + 5 = 0, \quad -\tfrac{1}{2}y_1 + y_2 + \tfrac{1}{5}x_2y_1 + x_2y_3 - \tfrac{5}{2} = 0, \\ & x_3y_2 - 9 = 0. \end{aligned}$$

$$\Longleftrightarrow \quad \underbrace{\begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 & -2 \\ 0 & 10 & 1 & 0 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y_1 - 7 \\ y_2 - 3 \\ x_2y_1 + 35 \\ x_2y_3 - 10 \\ x_3y_2 - 9 \end{bmatrix}}_{\boldsymbol{h}(\boldsymbol{z})} = \mathbf{0}$$

- Hard to apply privious approach to this problem
- Hard to find C and $\boldsymbol{h}(\boldsymbol{z})$

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Sum-of-Square Polynomial (SOS Polynomial)

A polynomial $p(\boldsymbol{z})$ is an sum-of-square (SOS) polynomial if

$$p(\boldsymbol{z}) = \sum_{i=1}^{r} \{q_i(\boldsymbol{z})\}^2$$

for some polynomials $q_1(\boldsymbol{z}), \ldots, q_r(\boldsymbol{z})$.

- $\Sigma_d[\boldsymbol{z}] \quad := \quad \{$ SOS polynomials $p(\boldsymbol{z}) \mid$ degree-d or less $\}$

Matrix representation of SOS polynomials

$p(\boldsymbol{z})$ is a degree-$2d$ SOS polynomial $\iff \exists W \in \mathbb{S}_+^{\binom{n+d}{d}}$ satisfying

$$p(\boldsymbol{z}) = \boldsymbol{u}_d(\boldsymbol{z})^{\mathrm{T}} W \boldsymbol{u}_d(\boldsymbol{z})$$

for $\boldsymbol{u}_d(\boldsymbol{z})$ consisting of all monomials of degree-d or less.

Equivalent problem with squared constraints

Start from (P) with $C = I_K$:

$$\begin{aligned} \min_{\boldsymbol{z} \in \mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & h_k(\boldsymbol{z}) = 0, \quad k \in \{1, \dots, K\}. \end{aligned} \tag{P}$$

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$$\downarrow$$

$$\begin{aligned} \min_{\boldsymbol{z}\in\mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & \{h_k(\boldsymbol{z})\}^2 \leq 0, \quad k \in \{1, \ldots, K\}. \end{aligned} \tag{P^2}$$

Differences:

- LHS of $h_k(\boldsymbol{z}) = 0$ becomes an SOS polynomials
- Equality → Inequality

Level-2 Sum-of-squares (SOS) Relaxation

$$\begin{aligned} \min_{\boldsymbol{z}\in\mathbb{R}^n} \quad & \{\boldsymbol{u}_2(\boldsymbol{z})\}^{\mathrm{T}} Q \boldsymbol{u}_2(\boldsymbol{z}) \\ \text{s.t.} \quad & \{h_k(\boldsymbol{z})\}^2 \leq 0, \quad k \in \{1, \ldots, K\}. \end{aligned} \tag{P^2}$$

$$\downarrow$$

$$\begin{aligned} \max_{\rho, \boldsymbol{\lambda}} \quad & \rho \\ \text{s.t.} \quad & \underbrace{(\boldsymbol{u}_2)^{\mathrm{T}} Q \boldsymbol{u}_2 - \rho + \sum_{k=1}^{K} h_k(\boldsymbol{z})^2 \lambda_k(\boldsymbol{z})}_{=:\, q(\boldsymbol{z};\, \rho, \boldsymbol{\lambda})} \in \Sigma_4[\boldsymbol{z}], \\ & \lambda_k(\boldsymbol{z}) \in \Sigma_{4-2d_k}[\boldsymbol{z}], \quad k \in \{1, \ldots, K\}. \end{aligned}$$

where $\Sigma_d[\boldsymbol{z}] \;:=\; \{$ SOS polynomials $p(\boldsymbol{z}) \mid$ degree d or less $\}$

$d_k \;:=\;$ degree of $h_k(\boldsymbol{z})$

Domain space was reduced thanks to $h_k(\boldsymbol{z})^2 \leq 0$.

Ranks of Two Optimal Solutions

Let M^* be *a solution of Lasserre's relaxation*,
and W^* be the matrix representation of

$$(\boldsymbol{u}_2)^{\mathrm{T}} Q \boldsymbol{u}_2 - \rho^* + \sum_{k=1}^{K} h_k(\boldsymbol{z})^2 \lambda_k^*(\boldsymbol{z})$$

of *a solution of SOS relaxation*.

Complementarity Condition

There is a pair of solutions (M^*, W^*) satisfying

$$M^* W^* = O \quad \text{under the strong duality.}$$

For any W^*, there exists M^* satisfying Sylvester rank inequality:

$$\begin{aligned}\operatorname{rank}(M^*) + \operatorname{rank}(W^*) &\leq N - \underbrace{\operatorname{rank}(M^* W^*)}_{=0} \\ &= N.\end{aligned}$$

Tightness for the SOS Relaxation

Sufficient condition of tightness

The SOS relaxation is tight
$\Longleftarrow$ $\exists$ an optimal solution $q(\boldsymbol{z}; \rho^*, \boldsymbol{\lambda}^*)$ of the SOS relaxation satisfying

$$\operatorname{rank} W^* \geq N - 1.$$

Proof   There exists an optimal solution M^* of the Lasserre's relaxation:

$$\begin{aligned} \operatorname{rank}(M^*) &\leq N - \operatorname{rank}(W^*) \\ &\leq N - (N-1) \qquad = 1 \end{aligned}$$

It follows from $M^*_{NN} = 1$ that $\operatorname{rank}(M^*) = 1$. $\square$

What matrix Q must be chosen for $\operatorname{rank} W^* \geq N - 1$?

Existence of Q for tight

Proposition

There exist Q and $\Gamma \in \mathbb{S}_+^N$ satisfying

$$(\boldsymbol{u}_2)^{\mathrm{T}} Q \boldsymbol{u}_2 - (\boldsymbol{u}_2)_0^{\mathrm{T}} Q (\boldsymbol{u}_2)_0 = (\boldsymbol{u}_2)^{\mathrm{T}} \Gamma \boldsymbol{u}_2$$

$$\operatorname{rank} Q = \operatorname{rank} \Gamma = N - 1$$

where $(\boldsymbol{u}_2)_0 := \boldsymbol{u}_2(\boldsymbol{z}_0)$ and $\boldsymbol{z}_0$ is a solution of (P).

$\therefore$ for any $\rho \in \mathbb{R}$ and $\lambda_k(\boldsymbol{z}) \in \Sigma_{4-2d_k}[\boldsymbol{z}]$ being a feasible solution,

SOS polynomial:

$$q(\boldsymbol{z}; \rho, \boldsymbol{\lambda}) = \underbrace{\underset{\substack{\text{SOS poly.} \\ \operatorname{rank}\Gamma = N-1}}{\underbrace{(\boldsymbol{u}_2)^{\mathrm{T}} Q \boldsymbol{u}_2 - \rho}} + \underset{\text{SOS poly.}}{\underbrace{\sum_{k=1}^{K} h_k(\boldsymbol{z})^2 \lambda_k(\boldsymbol{z})}}}_{\operatorname{rank} W = N-1} .$$

Idea for Proposition

For ith element of $\boldsymbol{u}_2$, define

$$Q^i := \begin{array}{c} \\ \\ \\ i) \\ \\ \end{array}\left[\begin{array}{c|ccc} [(\boldsymbol{u}_2)_0]_i^2 & \mathbf{0}^{\mathrm{T}} & \overset{(i}{-[(\boldsymbol{u}_2)_0]_i} & \mathbf{0}^{\mathrm{T}} \\ \hline \mathbf{0} & O & & \\ -[(\boldsymbol{u}_2)_0]_i & & 1 & \\ \mathbf{0} & & & O \end{array}\right] \in \mathbb{S}^N.$$

$$\Longrightarrow \quad \operatorname{rank}(Q^i) = 1, \quad (\boldsymbol{u}_2)_0^{\mathrm{T}}\, Q^i\, (\boldsymbol{u}_2)_0 = 0.$$

Focus on $\overline{Q} := \sum_{k=2}^{N} Q^i$, then

- $\operatorname{rank}\left(\overline{Q}\right) = N - 1.$
- $(\boldsymbol{u}_2)^{\mathrm{T}}\overline{Q}\boldsymbol{u}_2 - (\boldsymbol{u}_2)_0^{\mathrm{T}}\,\overline{Q}\,(\boldsymbol{u}_2)_0$ is a SOS polynomial.
- $\overline{Q}$ is sparse. (called an arrowhead matrix)

Tightness Conditions

Remark 4.5[2]

There exists coefficient matrix Q on the objective function such that the SOS relaxation of (P^2) with $C = I_K$ is tight.

$\overline{Q}$ is one choice that satisfies the remark.

$$\Downarrow$$

Generalization of C to nonsingular

The same discussion and the existence of Q hold
when $C^{\mathrm{T}}C$ is nonsingular.

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Problems for finding Q

Problems

- $(\boldsymbol{u}_2)_0$ depends on the true solution $\boldsymbol{z}_0$
- so does $\overline{Q}$

$$\overline{Q} = \begin{bmatrix} \sum_{r=1}^{N} [(\boldsymbol{u}_2)_0]_r^2 & -[(\boldsymbol{u}_2)_0]_2 & \cdots & -[(\boldsymbol{u}_2)_0]_N \\ -[(\boldsymbol{u}_2)_0]_2 & 1 & & \\ \vdots & & \ddots & \\ -[(\boldsymbol{u}_2)_0]_N & & & 1 \end{bmatrix}$$

Program to construct Q

$$
\begin{aligned}
\text{find} \quad & Q \in \mathbb{S}_+^N,\ U \in \mathbb{S}_+^{KN} \\
\text{s.t.} \quad & (\boldsymbol{u}_2)^{\mathrm{T}} Q \boldsymbol{u}_2 = (\boldsymbol{h}(\boldsymbol{z}) \otimes \boldsymbol{u}_2)^{\mathrm{T}} U \left(\boldsymbol{h}(\boldsymbol{z}) \otimes \boldsymbol{u}_2\right), \\
& Q_{ii} = 1, \quad i \in \{2, \ldots, N\}, \\
& Q_{ij} = 0, \quad (i,j) \in \{2, \ldots, N\} \times \{2, \ldots, N\},\ i \neq j,
\end{aligned}
\qquad (\mathscr{S}_1)
$$

Kronecker product

- The first row and column are recovered from $\boldsymbol{h}(\boldsymbol{z})$.
- The solution Q^* must be an arrowhead matrix.
    - Bottom-right block must be the identity matrix.

Numerical Experiment

Objective

To evaluate the accuracy of output Q from $(\mathscr{S}_1)$

Environment

- Intel Core i9-12900K / 64GB
- Julia 1.9.2 / Mosek 10.0.2 on Windows 11

Error of estimation

Let $\boldsymbol{z}_0$ be a true solution. The error of $\boldsymbol{z}^*$ returned by $(\mathscr{S}_1)$ is

$$\mathrm{Er}(\boldsymbol{z}^*) := \frac{\|\boldsymbol{z}^* - \boldsymbol{z}_0\|_2}{\|\boldsymbol{z}_0\|_2}$$

Generated Instances

Assumption: $n + m = 10$

10 random bipartite graphs $\mathcal{G}_1, \ldots, \mathcal{G}_{10}$ $\times$ 10 random rank-1 matrices $\boldsymbol{x}_0 (\boldsymbol{y}_0)^{\mathrm{T}}, \ldots$

$\Longrightarrow$ 100 instances

- Each edge corresponds to a hint of rank-one completion.

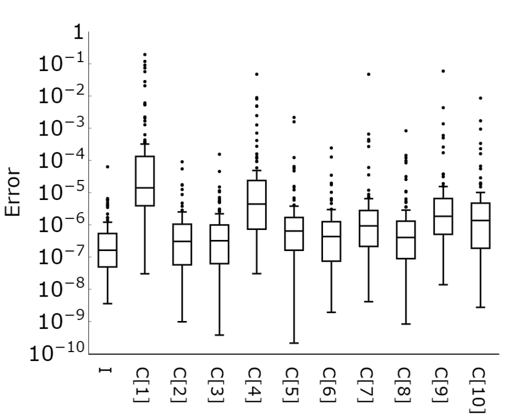
$$\boldsymbol{h}(\boldsymbol{z}) = \left[x_i y_j - (\boldsymbol{x}_0)_i \, (\boldsymbol{y}_0)_j \right]_{(i,j) \in E(\mathcal{G}_p)} = \mathbf{0}$$

- $\mathrm{Er}(\boldsymbol{z}^*)$ can be evaluated
  because the true solution is $\boldsymbol{z}_0 := \left[(\boldsymbol{x}_0)_2, \ldots, (\boldsymbol{x}_0)_m, (\boldsymbol{y}_0)^{\mathrm{T}} \right]^{\mathrm{T}}$.

- Random nonsingular matrices $C \in \mathbb{R}^{9 \times 9}$ in $C\boldsymbol{h}(\boldsymbol{z})$.

Results: Error of Constructed Q

- Exact solution of (P^2) can be recovered when $C = I$.
- Errors (and times) for nonsingular C_k is larger than $C = I$



Conclusion

Summary

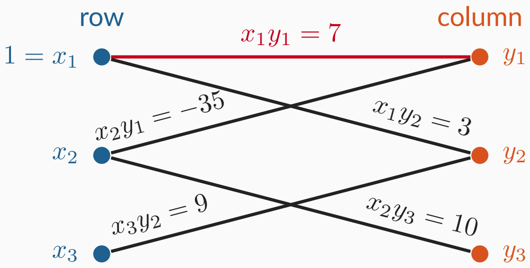
- We consider (P) in which rank-one matrix completion is hidden.
- Sparse and high-rank matrix Q controls the tightness of SOS relaxation.
- We provide an algorithm to find such a matrix Q.

Future works

- More detailed conditions of Q for tight
- Applying to rank-one tensor completion

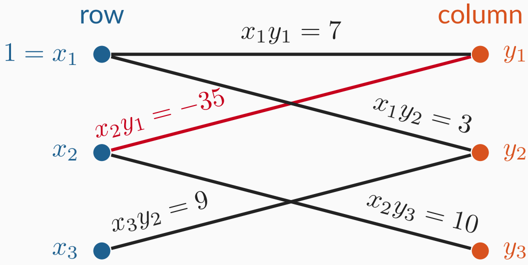
Thank you for your attention!
More information is available: arXiv:2311.14882.

Recovery using Bipartite Graph



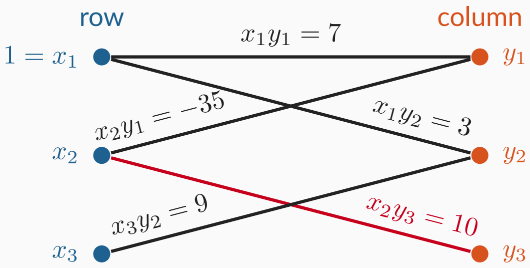
- $y_1 = 7$ from the edge $(1, 1)$,

Recovery using Bipartite Graph



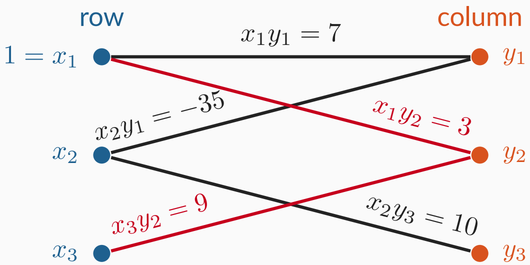
- $y_1 = 7$ from the edge $(1, 1)$,
- $x_2 = -5$ from the edge $(2, 1)$,

Recovery using Bipartite Graph



- $y_1 = 7$ from the edge $(1, 1)$,
- $x_2 = -5$ from the edge $(2, 1)$,
- $y_3 = -2$ from the edge $(2, 3)$,

Recovery using Bipartite Graph



- $y_1 = 7$ from the edge $(1, 1)$,
- $x_2 = -5$ from the edge $(2, 1)$,
- $y_3 = -2$ from the edge $(2, 3)$,
- $y_2 = 3$ and $x_3 = 3$ from the others.

Overview of Algorithm

Input:	constraints	$\boldsymbol{h}(\boldsymbol{z}) = 0$
Output:	estimated solution	$\boldsymbol{z}^*$

1: Solve ($\mathscr{S}_1$) and obtain a solution (Q^*, U^*).
2: $f_{\text{sum}} \leftarrow (\boldsymbol{u}_2)^{\mathrm{T}} Q^* \boldsymbol{u}_2$.
3: Solve ($\mathscr{S}_2(f_{\text{sum}})$) using f_{sum} and obtain a solution $(\rho^*, \boldsymbol{\lambda}^*)$.
4: Find the Gram matrix $\Gamma \in \mathbb{S}^N$ such that

$$(\boldsymbol{u}_2)^{\mathrm{T}} \Gamma \boldsymbol{u}_2 = f_{\text{sum}} - \rho^* + \sum_{k=1}^{K} \{h_k(\boldsymbol{z})\}^2 \lambda_k^*.$$

5: Find a vector $\boldsymbol{u}_2^* \in \mathbb{R}^N$ in the null space of Γ.
6: $\boldsymbol{z}^* \leftarrow \frac{1}{(\boldsymbol{u}_2^*)_1} \left[(\boldsymbol{u}_2^*)_2, (\boldsymbol{u}_2^*)_3, \ldots, (\boldsymbol{u}_2^*)_{s+1} \right]^{\mathrm{T}}$ and return $\boldsymbol{z}^*$.

Problem with Q^* in Second Stage

$$f_{\mathrm{sum}}(\boldsymbol{z}) \coloneqq (\boldsymbol{u}_2)^{\mathrm{T}} Q^* \boldsymbol{u}_2$$

$$\begin{aligned}
\max_{\rho,\boldsymbol{\lambda},\Delta_1,\ldots,\Delta_K} \quad & \rho \\
\text{s.t.} \quad & f_{\mathrm{sum}}(\boldsymbol{z}) - \rho + \sum_{k=1}^{K} h_k(\boldsymbol{z})^2 \lambda_k(\boldsymbol{z}) \in \Sigma_4[\boldsymbol{z}] \\
& \lambda_k(\boldsymbol{z}) = {\boldsymbol{u}_1}^{\mathrm{T}} \Delta_k \boldsymbol{u}_1 \quad (k = 1, \ldots, K) \\
& \mu I - \Delta_k \succeq O \quad (k = 1, \ldots, K) \\
& \rho \in \mathbb{R},\ \lambda_k(\boldsymbol{z}) \in \Sigma_{4-2d_k}[\boldsymbol{z}],\ \Delta_k \in \mathbb{S}_+^{m+n} \quad (k = 1, \ldots, K).
\end{aligned} \tag{$\mathscr{S}_2(f_{\mathrm{sum}})$}$$

Matrix C is NOT Identity

It becomes a problem in which
all equality constraints consist of linear combination of $\boldsymbol{h}(\boldsymbol{z})$

Example

$$\begin{cases} z_4 - 1.5z_5 - 2.5 = 0 \\ 5z_5 - 2z_3z_5 + 3 = 0 \\ 10z_5 + z_2z_4 + 5 = 0 \\ -0.5z_4 + z_5 + 0.2z_2z_4 + z_2z_6 - 2.5 = 0 \\ z_3z_5 - 9 = 0 \end{cases}$$

In fact,

$$\begin{aligned} & -0.5z_4 + z_5 + 0.2z_2z_4 + z_2z_6 - 2.5 \\ &= -0.5(z_4 - 7) + (z_5 - 3) + 0.2(z_2z_4 + 35) + (z_2z_6 - 10) \\ &= -0.5\, h_1(\boldsymbol{z}) + h_2(\boldsymbol{z}) + 0.2\, h_3(\boldsymbol{z}) + \qquad h_4(\boldsymbol{z}) \end{aligned}$$

combination of $h_1, \ldots, h_5$

Experiment

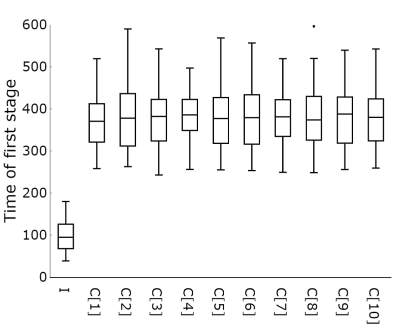
Experiment 1: Applying to Example Problem

Second Stage ($\mathscr{S}_2(f_{\mathrm{sum}})$)

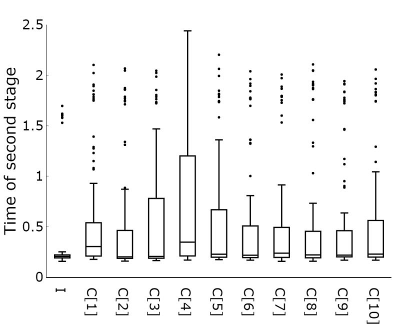
$$
\begin{aligned}
\max_{\rho,\boldsymbol{\lambda},\Delta_1,\Delta_2} \quad & \rho \\
\text{s.t.} \quad & f_{\mathrm{sum}} - \rho + (y_1-7)^2\lambda_1 + (y_2-3)^2\lambda_2 + (x_2y_1+35)^2\lambda_3 \\
& \quad +(x_2y_3-10)^2\lambda_4 + (x_3y_2-9)^2\lambda_5 \ \in \Sigma_4[\boldsymbol{z}] \\
& \lambda_k(\boldsymbol{z}) = {\boldsymbol{u}_1}^{\mathrm{T}}\Delta_k\boldsymbol{u}_1, \qquad k=1,2 \\
& \Delta_k \in \mathbb{S}_+^{n+m}, \ \ \mu I - \Delta_k \succeq O, \quad k=1,2 \\
& \rho \in \mathbb{R}, \ \ \lambda_3,\lambda_4,\lambda_5 \in [0,\mu]
\end{aligned}
$$

Time (s) ($\mathscr{S}_1$)	Time (s) ($\mathscr{S}_2(f_{\mathrm{sum}})$)	Optimal Value ($\mathscr{S}_2(f_{\mathrm{sum}})$)	Recovered Sol. $\boldsymbol{z}^*$	$\mathrm{Er}(\boldsymbol{z}^*)$
7.6	1.0	1.6×10^{-6}	$[-5.0; 3.0; 7.0; 3.0; -2.0]$	9.9×10^{-7}

Experiment 2: Caluculation Time



(a) Estimation of Q



(b) Solve by Q^*

"I": (P^2) with $C = I$, "$C[i]$": $C = C[i]$

Lagrange Function for (P^2)

Review: Problem with Squared Constraints

$$\begin{aligned} \min \quad & h_0(\boldsymbol{z}) \\ \text{s.t.} \quad & -h_k(\boldsymbol{z})^2 \geq 0, \quad k \in \{1, \ldots, K\}. \end{aligned} \tag{P^2}$$

Lagrange function

For $\boldsymbol{z} \in \mathbb{R}^{m+n-1}$ and $\lambda_k(\boldsymbol{z}) \in \Sigma[\boldsymbol{z}]$,

$$\mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}) := h_0(\boldsymbol{\lambda}) + \sum_{k=1}^{K} \lambda_k(\boldsymbol{z}) h_k(\boldsymbol{z})^2$$

Lagrange Dual Problem

Let $\mathcal{L}^*(\boldsymbol{\lambda}) := \inf \left\{ \mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}) \;\middle|\; \boldsymbol{z} \in \mathbb{R}^{m+n-1} \right\}$.

Lagrange Dual Problem

$$\begin{aligned} \max \quad & \mathcal{L}^*(\boldsymbol{\lambda}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \in (\Sigma[\boldsymbol{z}])^K . \end{aligned}$$

- $(\Sigma[\boldsymbol{z}]) \to (\Sigma_{2d-2d_k}[\boldsymbol{z}])$: Subproblem for Level-$d$

Lagrange Dual Problem

Let $\mathcal{L}^*(\boldsymbol{\lambda}) := \inf\left\{\mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}) \;\middle|\; \boldsymbol{z} \in \mathbb{R}^{m+n-1}\right\}$.

Lagrange Dual Problem

$$\begin{aligned} \max \quad & \mathcal{L}^*(\boldsymbol{\lambda}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \in \left(\Sigma[\boldsymbol{z}]_{2d-2d_k}\right)^K . \end{aligned}$$

- $(\Sigma[\boldsymbol{z}]) \rightarrow (\Sigma_{2d-2d_k}[\boldsymbol{z}])$: Subproblem for Level-$d$

Property of Generated Bipartite Graph

Table 1: Bipartite graphs.

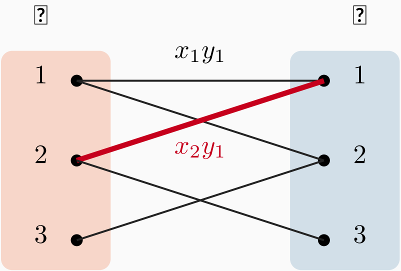
	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$	$\mathcal{G}_6$	$\mathcal{G}_7$	$\mathcal{G}_8$	$\mathcal{G}_9$	$\mathcal{G}_{10}$
n	9	7	6	2	6	2	1	8	1	6
m	1	3	4	8	4	8	9	2	9	4
max. degree	9	5	3	5	4	5	9	6	9	4

Graph Formulation for Γ

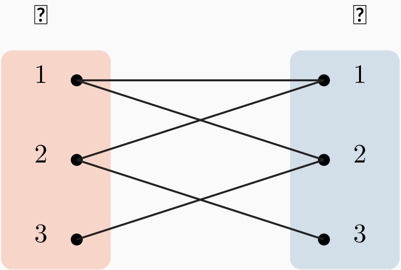
Bipartite $\mathcal{G}(\{1,\ldots,n\},\{1,\ldots,m\},\Gamma)$

row index   col index

Example $\{(1,1),(1,2),(2,1),(2,3),(3,2)\}$



Chain Structure and Unique Completion



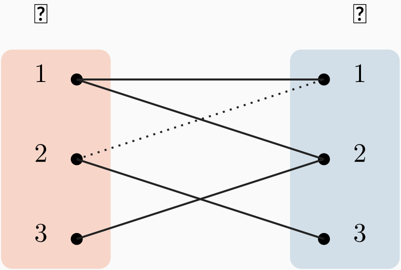
Fact

The graph is connected[2] $\iff$ Rank-one matrix completion is unique

- $x_1 \longrightarrow y_1 \longrightarrow x_2 \longrightarrow y_3$
- $x_1 \longrightarrow y_2 \longrightarrow x_3$

[2] a path exists between any two vertices

If the graph is DISconnected



$$\begin{bmatrix} 7 & 3 & ? \\ ? & ? & 10 \\ ? & 9 & ? \end{bmatrix} \to \begin{bmatrix} 7 & 3 & ? \\ ? & ? & 10 \\ \color{red}{21} & 9 & ? \end{bmatrix} \to ?$$

Solution

1. Divide the problem to problems of each connected component
2. Reorder numbers of indices on each problem
3. Solve them
4. Overlap them while comparing corresponding elements